THE STATIC EXTENSION PROBLEM IN GENERAL RELATIVITY

MICHAEL T. ANDERSON AND MARCUS A. KHURI

Abstract. We prove the existence of asymptotically flat solutions to the static vacuum Einstein equations on $M = \mathbb{R}^3 \setminus B$ with prescribed metric γ and mean curvature H on $\partial M \simeq S^2$, provided $H > 0$ and H has no critical points where the Gauss curvature $K_\gamma \le 0$. This gives a partial resolution of a conjecture of Bartnik on such static vacuum extensions. The existence and uniqueness of such extensions is closely related to Bartnik's definition of quasi-local mass.

1. Introduction

This paper is concerned with a conjecture of R. Bartnik [B3], [B4] on the existence and uniqueness of static solutions to the vacuum Einstein equations with certain prescribed boundary data. To describe the conjecture more precisely, let M be a 3-manifold diffeomorphic to $\mathbb{R}^3 \setminus B$ where B is a 3-ball, so that $\partial M \simeq S^2$. The static vacuum Einstein equations are the equations for a pair (g, u) consisting of a smooth Riemannian metric g on M and a positive potential function $u : M \to \mathbb{R}^+$ given by

$$uRic_g = D^2 u, \quad \Delta u = 0, \tag{1.1}$$

where the Hessian D^2 and Laplacian $\Delta = trD^2$ are taken with respect to g. The equations (1.1) are equivalent to the statement that the 4-dimensional metric

$$g_N = \pm u^2 dt^2 + g, \tag{1.2}$$

on the 4-manifold $N = \mathbb{R} \times M$ is Ricci-flat, i.e.

$$Ric_{g_N} = 0. \tag{1.3}$$

This holds for either choice of sign in (1.2) and since most of the analysis of the paper concerns the Riemannian data (g, u) in (1.1), we will assume g_N is Riemannian, and moreover identify t in (1.2) periodically, to obtain a metric on $N = S^1 \times M$ with t replaced by the angular variable θ.

Given (M, g, u) as above, let γ be the Riemannian metric induced on $S^2 = \partial M$ and let H be the mean curvature of $\partial M \subset (M, g)$, (with respect to the inward unit normal into M). Then the Bartnik conjecture [B4] states that, given an arbitrary such pair in C^∞,

$$(\gamma, H) \in Met^\infty(S^2) \times C^\infty_+(S^2), \quad H > 0, \tag{1.4}$$

there exists a unique asymptotically flat solution (g, u) to the static vacuum Einstein equations (1.1) inducing the boundary data (γ, H) on ∂M.

This conjecture is a natural outgrowth of Bartnik's concept of quasi-local mass $m_B(\Omega)$, [B2], [B3], defined as follows. Let (Ω, g) be a smooth compact 3-manifold with smooth boundary of non-negative scalar curvature, and define an *admissible extension* of (Ω, g) to be a complete, asymptotically flat 3-manifold $(\widetilde{M}, g)$ of non-negative scalar curvature in which (Ω, g) embeds isometrically and is not enclosed by any compact minimal surfaces (horizons). Then

$$m_B(\Omega) = \inf\{m_{ADM}(\widetilde{M}) : (\widetilde{M}, g) \text{ is an admissible extension of } (\Omega, g)\}, \tag{1.5}$$

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where $m_{ADM}(\widetilde{M})$ is the ADM mass of $(\widetilde{M}, g)$, cf. [B1]. Conjecturally, an extension $(\widetilde{M}, g)$ realizing the infimum in (1.5) is a solution to the static vacuum Einstein equations (1.1) on $M = \widetilde{M} \setminus \Omega$ which is Lipschitz, (but not smooth), across the junction $\partial\Omega$ and for which the induced metric and mean curvature at the boundary of the interior and exterior regions agree:

$$g|_{\partial M} = g|_{\partial\Omega}, \quad H_{\partial M} = H_{\partial\Omega},$$

leading to the boundary data (1.4). Observe that the boundary data (γ, H) have the character of a mixed Dirichlet-Neumann type boundary value problem for the static equations (1.1), but the potential function u is absent from the boundary data. We point out that more standard Dirichlet or Neumann boundary data are not suitable for the (static) Einstein equations, cf. [An3].

In this paper, we prove the existence part of the Bartnik conjecture at least for a large class of boundary data (1.4). Thus, let $Met^{m,\alpha}(\partial M)$ be the space of $C^{m,\alpha}$ metrics on $\partial M \simeq S^2$ and $C_+^{m-1,\alpha}(\partial M)$ be the space of $C^{m-1,\alpha}$ positive functions on ∂M. Let K_γ be the Gauss curvature of a metric $\gamma \in Met^{m,\alpha}(\partial M)$.

Theorem 1.1. *For any pair $(\gamma, H) \in Met^{m,\alpha}(\partial M) \times C_+^{m-1,\alpha}(\partial M)$, $m \geq 3$, satisfying the non-degeneracy condition*

$$dH \neq 0 \quad \text{on} \quad \{K_\gamma \leq 0\}, \tag{1.6}$$

there exists an asymptotically flat solution (M, g) to the static vacuum Einstein equations (1.1) *on $M \simeq \mathbb{R}^3 \setminus B$, realizing the data (γ, H) at ∂M.*

The condition (1.6) means that the mean curvature H has no critical points where $K_\gamma \leq 0$. In particular it holds for all metrics γ of positive Gauss curvature on ∂M.

To describe the approach taken towards the proof of Theorem 1.1, let $\mathcal{E}_S = \mathcal{E}_S^{m,\alpha}$ be the moduli space of AF static vacuum solutions (M, g, u) on a given 3-manifold M which are $C^{m,\alpha}$ up to ∂M, $m \geq 3$, and let

$$\Pi_B : \mathcal{E}_S^{m,\alpha} \to Met^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \tag{1.7}$$

$$\Pi_B(g) = (\gamma, H),$$

be the Bartnik boundary map. We prove in Theorem 3.6 below that the space $\mathcal{E}_S$ is a smooth (infinite dimensional) Banach manifold, and Π_B is smooth and Fredholm, of Fredholm index 0.

Let $\mathcal{E}^+$ be the open submanifold of $\mathcal{E}_S$ for which the mean curvature H is positive, i.e.

$$\mathcal{E}^+ = (\Pi_B)^{-1}(Met^{m,\alpha}(\partial M) \times C_+^{m-1,\alpha}(\partial M)).$$

The Bartnik conjecture above may thus be rephrased to state that the smooth map Π_B, restricted to $\mathcal{E}^+$,

$$\Pi_B : \mathcal{E}^+ \to Met^{m,\alpha}(\partial M) \times C_+^{m-1,\alpha}(\partial M), \tag{1.8}$$

is surjective and injective, i.e. essentially a smooth diffeomorphism.

A basic issue in the proof of the Bartnik conjecture is whether the map Π_B on $\mathcal{E}^+$ is proper. As shown below, this is not true in general. However, let $\mathcal{B}^{nd} \subset Met^{m,\alpha}(\partial M) \times C_+^{m-1,\alpha}(\partial M)$ be the open subset on which the non-degeneracy condition (1.6) holds. Also let

$$\mathcal{E}^{nd} = (\Pi_B)^{-1}(\mathcal{B}^{nd}).$$

It is not known if the spaces $\mathcal{E}^+$, $\mathcal{E}^{nd}$ or $\mathcal{B}^{nd}$ are connected, so throughout the paper we assume that $\mathcal{E}^+$, $\mathcal{E}^{nd}$ and $\mathcal{B}^{nd}$ are the connected components of these Banach manifolds containing the standard flat solution $(M, g_{flat}, 1)$ where g_{flat} is the flat metric on $\mathbb{R}^3 \setminus B^3(1)$ with $u \equiv 1$. The boundary data of this solution are thus $(\gamma_{+1}, 2)$, where γ_{+1} is the round metric on the 2-sphere of radius 1.

One of the main results of the paper, cf. Corollary 5.3, is then that the map Π_B restricted to $\mathcal{E}^{nd}$ is proper, i.e.

$$\Pi_B : \mathcal{E}^{nd} \to \mathcal{B}^{nd}, \tag{1.9}$$

is a smooth proper Fredholm map. Such Fredholm maps have a mod 2 degree, $deg_{\mathbb{Z}_2}\Pi_B$, cf. [Sm]. Using the black hole uniqueness theorem [I],[R],[BM], we then prove in Theorem 6.1 that

$$deg_{\mathbb{Z}_2}\Pi_B = 1,$$

on $\mathcal{E}^{nd}$. This proves Theorem 1.1, since a map of non-zero degree is necessarily surjective.

We point out that the map Π_B in (1.8) on the full space $\mathcal{E}^+$ is not proper. For example, for the flat solution g_{flat} with $u = 1$, there are boundaries given by embedded spheres (S^2, γ_i, H_i) with H_i uniformly positive, which converge smoothly in $\mathbb{R}^3$ to a limit which is an immersed but not embedded sphere. Such a limit is then in the boundary $\partial\mathcal{E}^+$, but the limit boundary data $(\gamma, H) \in Met^{m,\alpha}(\partial M) \times C_+^{m-1,\alpha}(\partial M)$; cf. Remark 4.3 for further discussion. This implies of course that the full boundary map Π_B in (1.8) cannot be a diffeomorphism.

In Theorem 5.1, the properness of the boundary map Π_B is established for boundaries $\partial M \subset (M, g)$ which are *outer-minimizing*, i.e. for which

$$area(\Sigma) \geq area(\partial M), \tag{1.10}$$

for any surface $\Sigma \subset M$ homologous to ∂M. (In fact, properness holds for solutions for which (1.10) holds in an arbitrarily small but fixed neighborhood of ∂M). However, (1.10) is not a property of the boundary data (γ, H), since it involves the (global) structure of the solution (M, g, u).

To tackle this problem, in Theorem 4.1 we prove that any static vacuum solution $(M, g, u) \in \mathcal{E}^{nd}$ has a global foliation by surfaces $\Sigma_t \simeq S^2$ with positive mean curvature and with $\Sigma_0 = \partial M$. This is shown by proving that such solutions admit (a small perturbation of) a global inverse mean curvature flow (IMCF) starting at ∂M; this is where the non-degeneracy condition (1.6) is needed. Manifolds admitting such a positive mean curvature foliation automatically satisfy (1.10).

Note that some condition, such as (1.6), is needed to establish the outer-minimizing property. If (1.10) held for all solutions in $\mathcal{E}^+$, then Π_B in (1.8) would be proper, which we know is not the case.

Finally, it is worth mentioning that besides its relevance to general relativity, Theorem 1.1 appears to be the first global (or semi-global) existence result for solutions of the Einstein equations on domains with a compact boundary component, in dimensions ≥ 4.

The contents of the paper are briefly as follows. In §2, we present background information on the structure of static vacuum solutions and choices of gauge. Section 3 discusses elliptic boundary value problems for the Einstein equations and proves the basic structure theorems on the moduli space $\mathcal{E}_S$ and the boundary map Π_B. In §4, we prove that solutions in $\mathcal{E}^{nd}$ have a global positive mean curvature foliation. In §5, we then prove the requisite apriori estimates and establish the properness of Π_B on $\mathcal{E}^{nd}$. Finally, §6 contains the computation of the degree of Π_B and closes with several related remarks and open questions.

2. Background Discussion

Let M be a 3-manifold with compact boundary ∂M, and with a single open end E. (All of the results of this section and of §3 hold in all dimensions, but for simplicity, we restrict to dimension 3). Apriori, ∂M need not be connected, although this will be assumed later on. As following (1.2)-(1.3), we let $N = S^1 \times M$.

Let $Met_S(N) = Met_S^{m,\alpha}(N)$ be the space of complete $C^{m,\alpha}$ static metrics on N, i.e. metrics of the form (1.1), $m \geq 2$. One has

$$Met_S^{m,\alpha}(N) \simeq Met^{m,\alpha}(M) \times C_+^{m,\alpha}(M), \tag{2.1}$$

where $C_+^{m,\alpha}(M)$ is the space of positive $C^{m,\alpha}$ functions on M. The space $\mathbb{E}_S = \mathbb{E}_S(N)$ of static Einstein (Ricci-flat) metrics on N is equivalent to the space of pairs $g_N = (g_M, u) \in Met(M) \times C_+(M)$ satisfying (1.1) or (1.3).(The smoothness indices will be occasionally dropped when unimportant).

Recall that a complete metric $g \in Met^{m,\alpha}(E)$ on an end E is asymptotically flat if E is diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is a 3-ball, and there exists a diffeomorphism $F : \mathbb{R}^3 \setminus B \to E$ such that, in the chart F,

$$g_{ij} = \delta_{ij} + O(r^{-1}), \quad \partial_k g_{ij} = O(r^{-2}), \quad \partial_k \partial_\ell g_{ij} = O(r^{-3}), \tag{2.2}$$

in the standard Euclidean coordinates on $\mathbb{R}^3$. The static vacuum equations (1.1) are invariant under multiplication of the potential u by constants. Throughout the paper, we assume that u is normalized so that $u \to 1$ at infinity, and that u is asymptotically constant in the sense that

$$u = 1 + O(r^{-1}), \quad \partial_k u = O(r^{-2}), \quad \partial_k \partial_\ell u = O(r^{-3}). \tag{2.3}$$

Thus the 4-metric g_N is asymptotic to the product $S^1 \times \mathbb{R}^3$ at infinity.

It is proved in [An2] that ends of static vacuum solutions (M, g, u) are either asymptotically flat or parabolic, where parabolic is understood in the sense of potential theory; equivalently, parabolic ends have small volume growth in that the area of geodesic spheres grows slower than $r^{1+\varepsilon}$, for any $\varepsilon > 0$. Moreover, asymptotically flat ends are strongly asymptotically flat in that the metric and potential have asymptotic expansions of the form

$$g_{ij} = (1 + \frac{2m}{r})\delta_{ij} + \cdots, \quad u = 1 - \frac{2m}{r} + \cdots, \tag{2.4}$$

where the mass m may apriori be any value $m \in \mathbb{R}$. These two behaviors, asymptotically flat and parabolic, are radically different and there is no curve of asymptotic structures for static vacuum solutions which join them. The finer behavior of asymptotically flat ends are described by the mass parameter m and higher multipole moments, cf. [BS].

Let $\widetilde{g}$ be a fixed asymptotically flat (background) metric in $\mathbb{E}_S$; henceforth $\mathbb{E}_S$ will denote the space of asymptotically flat static vacuum Einstein solutions. The static Einstein equations are not elliptic, due to their invariance under diffeomorphisms, and for several reasons one needs to choose an elliptic gauge. To begin, we consider the Bianchi gauge, and define

$$\Phi_{\widetilde{g}} : Met_S^{m,\alpha}(N) \to S^{m-2,\alpha}(N), \tag{2.5}$$

$$\Phi_{\widetilde{g}}(g) = Ric_g + \delta_g^* \beta_{\widetilde{g}}(g),$$

where $\beta_{\widetilde{g}}$ is the Bianchi operator, $\beta_{\widetilde{g}}(g) = \delta_{\widetilde{g}}(g) + \frac{1}{2} dtr_{\widetilde{g}}(g)$. Also, $(\delta^* X)(A, B) = \frac{1}{2}(\langle \nabla_A X, B \rangle + \langle \nabla_B X, A \rangle)$ and $\delta X = -tr(\delta^* X)$ is the divergence. The operator $\Phi_{\widetilde{g}}$ is a C^∞ smooth map into the space $S^{m-2,\alpha}(N)$ of static symmetric bilinear forms on N.

Using standard formulas for the linearization of the Ricci and scalar curvatures, cf. [Be] for instance, the linearization of Φ at $g = \widetilde{g} \in \mathbb{E}_S$ is given by

$$L(h) = 2(D\Phi_g)(h) = D^* D h - 2R(h). \tag{2.6}$$

Clearly, L is elliptic and formally self-adjoint. In §3 we will discuss boundary value problems for Φ and L.

Next, the asymptotic behavior in the asymptotically flat end E requires the introduction of suitable weighted function spaces. We will use the standard weighted Hölder spaces, although one could equally well use weighted Sobolev spaces. Thus, define $Met_\delta^{m,\alpha}(N) \subset Met_S^{m,\alpha}(N)$ to be the subspace of metrics which decay to Euclidean data at a rate $r^{-\delta}$ at infinity; more precisely, the component functions g_{ij} and u of g_N should satisfy

$$g_{ij} - \delta_{ij} \in C_\delta^{m,\alpha}(\mathbb{R}^3 \setminus B), \quad u - 1 \in C_\delta^{m,\alpha}(\mathbb{R}^3 \setminus B).$$

Here C^m_δ consists of functions v such that

$$||v||_{C^m_\delta} = \sum_{k=0}^{m} \sup r^{k+\delta}|\nabla^k v| < \infty,$$

while $C^{m,\alpha}_\delta$ consists of functions such that

$$||v||_{C^{m,\alpha}_\delta} = ||v||_{C^m_\delta} + \sup_{x,y}[\min(r(x), r(y))^{m+\alpha+\delta}\frac{|\nabla^m v(x) - \nabla^m v(y)|}{|x-y|^\alpha}] < \infty,$$

cf. [B1], [LP]. Throughout the following, we assume the decay rate δ is fixed, and chosen to satisfy

$$\tfrac{1}{2} < \delta < 1. \tag{2.7}$$

It is well-known, cf. [B1], [LP], that the Laplacian on functions, and Laplace-type operators on tensors, as in (2.6), are Fredholm when acting on these weighted Hölder spaces.

The map Φ in (2.5) clearly induces a smooth map

$$\Phi : Met^{m,\alpha}_\delta(N) \to S^{m-2,\alpha}_\delta(N). \tag{2.8}$$

Observe that g is Einstein if $\Phi_{\widetilde{g}}(g) = 0$ and $\beta_{\widetilde{g}}(g) = 0$, so that g is in Bianchi gauge with respect to $\widetilde{g}$. (Although $\Phi_{\widetilde{g}}$ is defined for all $g \in Met^{m,\alpha}_\delta(N)$, we will only consider it acting on g near $\widetilde{g}$).

Given $\widetilde{g} \in \mathbb{E}_S$, let $Met^{m,\alpha}_C(N) \subset Met^{m,\alpha}_\delta(N)$ be the space of $C^{m,\alpha}$ smooth static AF Riemannian metrics on N, satisfying the Bianchi gauge constraint

$$\beta_{\widetilde{g}}(g) = 0 \ \text{ at } \ \partial N. \tag{2.9}$$

As above,

$$\Phi : Met^{m,\alpha}_C(N) \to S^{m-2,\alpha}_\delta(N).$$

Similarly let $Z^{m,\alpha}_C$ be the space of metrics $g \in Met^{m,\alpha}_C(N)$ satisfying $\Phi_{\widetilde{g}}(g) = 0$, and let

$$\mathbb{E}_C \subset Z_C \tag{2.10}$$

be the subset of static Einstein metrics $g = g_N$, $Ric_g = 0$ in Z_C. The following result justifies the use of the operator Φ to study $\mathbb{E}_C$.

Proposition 2.1. *Any static metric $g = g_N \in Z_C$ sufficiently close to $\widetilde{g}$ is necessarily Einstein, $g \in \mathbb{E}_C$. Moreover, this also holds infinitesimally in the following sense. Let κ be an infinitesimal deformation of $g \in Z_C$, i.e. $\kappa \in KerD\Phi$. If $\beta_{\widetilde{g}}(g) = 0$, (for example $\widetilde{g} = g$), then*

$$\beta_{\widetilde{g}}(\kappa) = 0, \tag{2.11}$$

and κ is an infinitesimal Einstein deformation, i.e. the variation of g in the direction κ preserves (1.3) *to* 1^{st} *order.*

Proof: Since $g \in Z_C$, one has $\Phi(g) = 0$, i.e.

$$Ric_g + \delta^*_g\beta_{\widetilde{g}}(g) = 0.$$

Applying the Bianchi operator β_g and using the Bianchi identity $\beta_g(Ric_g) = 0$ gives

$$\beta_g(\delta^*_g(\beta_{\widetilde{g}}(g))) = 0. \tag{2.12}$$

Setting $V = \beta_{\widetilde{g}}(g)$, a standard Weitzenbock formula shows that $2\beta_g\delta^*_g(V) = D^*DV - Ric(V)$. Also, since $g, \widetilde{g} \in Met^{m,\alpha}_\delta(N)$, $V \in \chi^{m-1,\alpha}_{1+\delta}(M)$, where $\chi^{m-1,\alpha}_{1+\delta}(M)$ is the space of vector fields whose components are in $C^{m-1,\alpha}_{1+\delta}(M)$. When acting on vector fields V with $V = 0$ on ∂M, as in (2.9), the operator D^*D is positive, with trivial kernel. Namely, if $W \in C^{m-1,\alpha}_{1+\delta}$ is in the kernel of D^*D, then integrating by parts gives

$$\int_{B(r)} |DW|^2 + \int_{S(r)} \langle W, \nabla_N W\rangle = 0,$$

where $B(r) = \{x \in M : dist(x, \partial M) \leq r\}$. (Since $W = 0$ on ∂M, there is no boundary term at ∂M). Letting $r \to \infty$, the boundary integral tends to 0 and so $DW = 0$, which in turn implies $W = 0$.

Since D^*D is self-adjoint and Fredholm, it has a smallest positive eigenvalue bounded away from 0. For g sufficiently close to $\widetilde{g}$, $|Ric(V)| = O(\varepsilon)$ pointwise and $Ric(V) \in C^{m-3,\alpha}_{3+\delta}(M)$, so that we may assume that $2\beta_g\delta_g^*(V)$ is a positive operator on V. Hence, again since $V = 0$ on ∂M, the only solution of (2.12) is $V = 0$, which implies $g \in \mathbb{E}_C$.

To prove the second statement, let $g_t = g + t\kappa$. Applying the Bianchi operator β_{g_t} to $\Phi(g_t)$ gives

$$\beta_{g_t}\Phi(g_t) = \beta_{g_t}\delta^*_{g_t}(\beta_{\widetilde{g}}(g_t)). \tag{2.13}$$

Taking the derivative with respect to t at $t = 0$, one has $(\beta_{g_t}\Phi(g_t))' = \beta'\Phi + \beta\Phi'$. Both terms here vanish since $g \in Z_C$ and κ is formally tangent to Z_C. Hence the variation of the right hand side of (2.13) vanishes. Since $\beta_{\widetilde{g}}(g) = 0$, this gives $\beta_g\delta_g^*(\beta_{\widetilde{g}}(\kappa)) = 0$. The equation (2.11) then follows exactly as following (2.12), with $V = \beta_{\widetilde{g}}(\kappa)$.

■

Let $\mathcal{D}_1^{m+1,\alpha}$ denote the space of $C^{m+1,\alpha}_\delta$ static diffeomorphisms of N which equal the identity on ∂N. Thus such diffeomorphisms decay to the identity at the rate $r^{-\delta}$ and are independent of the t or θ-variable in (1.2). The group $\mathcal{D}_1$ acts freely and continuously on $Met(N)$ and $\mathbb{E}_S$ by pullback and one has the following local slice theorem for this action; we refer to [An3] for the proof.

Lemma 2.2. *Given any $\widetilde{g} \in \mathbb{E}_S^{m,\alpha}$ and $g \in Met_\delta^{m,\alpha}(N)$ near $\widetilde{g}$, there exists a unique diffeomorphism $\phi \in \mathcal{D}_1^{m+1,\alpha}$ close to the identity, such that*

$$\beta_{\widetilde{g}}(\phi^* g) = 0. \tag{2.14}$$

In particular, $\phi^* g \in Met_C^{m,\alpha}(N)$.

Lemma 2.2 implies that if $g \in \mathbb{E}_S^{m,\alpha}$ is a static Einstein metric near $\widetilde{g}$, then g is isometric, by a diffeomorphism in $\mathcal{D}_1^{m+1,\alpha}$, to an Einstein metric in $\mathbb{E}_C^{m,\alpha}$, so that $\mathbb{E}_C^{m,\alpha}$ is a slice for $\mathbb{E}_S^{m,\alpha}$ under the action of $\mathcal{D}_1^{m+1,\alpha}$.

To prove that the moduli space $\mathcal{E}$ is a smooth Banach manifold, (cf. Theorem 3.6), it is important to have a gauge with choice of boundary data in which the Einstein equations form a self-adjoint elliptic boundary value problem. This is not the case for the operator Φ and we are not aware of geometrically natural self-adjoint boundary conditions for Φ. For this reason, we will also consider another natural gauge, namely the divergence-free gauge.

To do this, instead of Φ, consider

$$\hat{\Phi}(g) = \hat{\Phi}_{\widetilde{g}}(g) = Ric_g - \frac{s}{2}g + \delta_g^*\delta_{\widetilde{g}}g, \tag{2.15}$$

where s is the scalar curvature of $g = g_N$. The linearization of $\hat{\Phi}$ at $g = \widetilde{g} \in \mathbb{E}_S$ is given by

$$\hat{L}(h) = 2(D\hat{\Phi}_{\widetilde{g}})_g(h) = D^*Dh - 2R(h) - (D^2 trh + \delta\delta h\, g) + \Delta trh\, g. \tag{2.16}$$

In analogy to (2.9), define

$$Met_D^{m,\alpha}(M) = \{g \in Met_\delta^{m,\alpha}(N) : \delta_{\widetilde{g}}g = 0 \ \text{ at } \ \partial N\}. \tag{2.17}$$

Similarly, let $Z_D^{m,\alpha} = \hat{\Phi}^{-1}(0) \cap Met_D^{m,\alpha}(M)$ and $\mathbb{E}_D \subset Z_D$ be the space of Einstein metrics in divergence-free gauge with respect to $\widetilde{g} \in \mathbb{E}_S$.

It is easy to see that Proposition 2.1 and Lemma 2.2 hold in this divergence-free gauge in place of the previous Bianchi gauge, with essentially the same proof. Thus $\mathbb{E}_D = Z_D$ and for $g \in \mathbb{E}_D$,

$$\delta_{\widetilde{g}}g = 0 \ \text{ on } \ N. \tag{2.18}$$

Moreover, the diffeomorphism group $\mathcal{D}_1$ transforms one gauge choice uniquely to the other. For instance, suppose $\beta_{\widetilde{g}}(g) = 0$. Then we claim there is a unique $\phi \in \mathcal{D}_1^{m+1,\alpha}$ such that

$$\delta_{\widetilde{g}}(\phi^* g) = 0. \tag{2.19}$$

At the linearized level, with $g = \widetilde{g}$, this amounts to finding a vector field V with $V = 0$ on ∂N such that if $\beta_{\widetilde{g}} h = 0$ then $\delta_{\widetilde{g}}(h + \delta^* V) = 0$. This equation is equivalent to the equation $\delta\delta^* V = \frac{1}{2} dtrh$, which is uniquely solvable for V with $V = 0$ on ∂N. The local result in (2.19) then follows from the inverse function theorem.

3. The Moduli Space

In this section, we study boundary value problems for the elliptic operators Φ and $\hat{\Phi}$, and use this to prove that the moduli space $\mathcal{E}_S$ of static vacuum solutions is a smooth Banach manifold for which the boundary map Π_B is Fredholm, of Fredholm index 0, cf. Theorem 3.6.

We begin with the Bianchi-gauged Einstein operator Φ in (2.5), i.e.

$$\Phi_{\widetilde{g}}(g) = Ric_g + \delta_g^* \beta_{\widetilde{g}}(g).$$

Let A denote the 2^{nd} fundamental form of ∂M in M, $A(X,Y) = \langle \nabla_X N, Y \rangle$, where N is the unit inward normal into M, X, Y tangent to ∂M. Similarly, let $H_M = trA$ denote the mean curvature of ∂M in M. Throughout the paper W^T will denote the restriction or the orthogonal projection of a tensor W to $T(\partial N)$ or $T(\partial M)$.

Proposition 3.1. *Near any given background solution $\widetilde{g} \in \mathbb{E}_S^{m,\alpha}$, the operator $\Phi = \Phi_{\widetilde{g}}$ in* (2.5) *with boundary conditions:*

$$\beta_{\widetilde{g}}(g) = 0, \quad g|_{T(\partial M)} = \gamma_M, \quad H_M = h \quad at \quad \partial N, \tag{3.1}$$

is an elliptic boundary value problem of Fredholm index 0.

Here the induced metric γ_M is in $Met^{m,\alpha}(\partial M)$ while the mean curvature H_M of ∂M in (M, g_M) is in $C^{m-1,\alpha}(\partial M)$. Note that the potential u does not enter this boundary data and so is formally undetermined at ∂M. Also the static property implies that $\beta_{\widetilde{g}}(g)$ vanishes in the vertical direction, $\beta_{\widetilde{g}}(g)(\partial_\theta) = 0$.

Proof: It suffices to prove that the leading order part of the linearized operators forms an elliptic system. Recall from (2.6) that the linearization of Φ at $\widetilde{g} = g$ is given by

$$L(h) = 2(D\Phi_g)(h) = D^* D h - 2R(h).$$

The leading order symbol of $L = 2D\Phi$ at ξ' is

$$\sigma(L) = -|\xi'|^2 I, \tag{3.2}$$

where I is the $Q \times Q$ identity matrix, with $Q = (n(n+1)/2) + 1$; Q is the sum of the dimension of the space of symmetric bilinear forms on $\mathbb{R}^n$, together with the extra vertical S^1 direction. Here $n = 3$ but we give the proof for general dimensions. For static metrics, all components of the metric are locally functions on $\mathbb{R}^n$, and all derivatives in the vertical S^1 direction are trivial. In the following, the subscript 0 represents the direction normal to ∂M in M, (or ∂N in N), subscript 1 denotes the vertical direction, tangent to S^1, while indices 2 through n represent the directions tangent to ∂M. Note that one has $h_{1\alpha} = 0$, for all $\alpha \neq 1$. The positive roots of (3.2) are $i|\xi|$, where $\xi' = (\xi_0, \xi)$, with multiplicity Q at $\xi \in T^*(\mathbb{R}^n)$.

Writing $\xi' = (z, \xi_i)$, $i = 2, \dots, n$, (as above $\xi_1 = 0$), the symbols of the leading order terms in the boundary operators are given by:

$$-2izh_{0k} - 2i \sum_{j \geq 2} \xi_j h_{jk} + i\xi_k trh = 0, \quad k \geq 2,$$

$$-2izh_{00} - 2i\sum_{k\geq 2}\xi_k h_{0k} + iztrh = 0,$$

$$h^T = (\gamma')^T, \ \ \text{and} \ \ (H_M)'_h = \omega.$$

This gives $n+\frac{n(n-1)}{2}+1 = Q$ boundary equations, as required. Ellipticity requires that the operator defined by the boundary symbols above has trivial kernel when z is set to the root $i|\xi|$. Carrying this out then gives the system

$$2|\xi|h_{0k} - 2i\sum_{j\geq 2}\xi_j h_{jk} + i\xi_k trh = 0, \quad k \geq 2, \tag{3.3}$$

$$2|\xi|h_{00} - 2i\sum_{k\geq 2}\xi_k h_{0k} - |\xi|trh = 0, \tag{3.4}$$

$$h_{11} = \phi, \quad h^T = 0, \quad (H_M)'_h = 0, \tag{3.5}$$

where ϕ is an undetermined function.

Multiplying (3.3) by $i\xi_k$ and summing gives, via (3.5),

$$2|\xi|i\sum_{k\geq 2}\xi_k h_{0k} = |\xi|^2 trh.$$

Substituting (3.4) on the term on the left above then gives

$$2|\xi|^2 h_{00} - 2|\xi|^2 trh = 0.$$

Since $trh = h_{00} + \phi$, it follows that $\phi = 0$.

Next, to compute H'_M, we first observe that in general

$$2A'_h = \nabla_N h + 2A \circ h - 2\delta^*(h(N)^T) - \delta^*(h_{00}N). \tag{3.6}$$

This follows by differentiating the defining formula $2A = \mathcal{L}_N g$, and using the identities $2N'_h = -2h(N)^T - h_{00}N$, $\mathcal{L}_N h = \nabla_N h + 2A \circ h$. Since $H_M = tr_M A$, $H'_h = (tr_M)'_h A - tr_M A \circ h$ and so

$$2(H_M)'_h = tr_M(\nabla_N h - 2\delta^*(h(N)^T) - \delta^*(h_{00}N)). \tag{3.7}$$

Hence the symbol of $2(H_M)'_h$ is given by $\sum_{k\geq 2}(izh_{kk} - 2i\xi_k h_{0k})$. Setting this to 0 at the root $z = i|\xi|$ gives

$$\sum_{k\geq 2}(|\xi|h_{kk} + 2i\xi_k h_{0k}) = 0. \tag{3.8}$$

Via (3.5), this gives $-2i\sum_{k\geq 2}\xi_k h_{0k} = 0$, and substituting this in (3.4) and using the fact that $\phi = 0$ gives

$$2|\xi|h_{00} - |\xi|h_{00} = 0,$$

so that $h_{00} = 0$. It follows from (3.3) that $h_{0k} = 0$ and hence $h = 0$. This proves ellipticity of the boundary value problem (3.1) and the Fredholm property follows from the fact that the Laplace-type operator L is Fredholm on $Met_\delta^{m,\alpha}$, cf. [LP].

Finally, it is straightforward to verify that the boundary data (3.1) may be continuously deformed through elliptic boundary data to elliptic boundary data for which L is self-adjoint and so of index 0. This is proved in [An3] in a slightly different setting and the proof carries over here with only minor change, and so we refer to [An3] for further details. The homotopy invariance of the index then completes the proof.

■

As noted in §2, we are not aware of a geometrically natural self-adjoint elliptic boundary value problem for Φ. In particular, the boundary conditions (3.1) are not self-adjoint. This property is

important for the proof of Theorem 3.6, and for this reason, we turn to the operator $\hat{\Phi}$ in (2.15) with linearization at $\widetilde{g} = g$ given by $\hat{L}$ in (2.16).

Regarding boundary conditions for $\hat{L}$, for $h \in S_{\delta}^{m-2,\alpha}(N)$, let $h^T = h|_{\partial N}$ and $[h^T]_0$ be the projection of h^T onto the space of forms trace-free with respect to $\gamma = \gamma_N$. Similarly, H'_h denotes here the linearization of the mean curvature $H = H_N$ of $\partial N \subset N$.

We then have:

Lemma 3.2. *The operator $\hat{L}$ with boundary conditions*

$$\delta h = 0, \quad [h^T]_0 = 0, \quad H'_h = 0, \tag{3.9}$$

is a self-adjoint elliptic operator. Moreover, under the first two conditions $\delta h = 0$ and $[h^T]_0 = 0$, the operator L is self-adjoint exactly for the boundary condition $H'_h = 0$.

Proof: It is a rather long (and uninteresting) calculation to prove that the operator $\hat{L}$ with boundary data (3.9) forms an elliptic system; this has been verified by computer computation using Maple. More conceptually, instead we will make use of Proposition 3.1 to simplify the proof. First, recall, [ADN], [Tr], that ellipticity of a boundary value problem is equivalent to the existence of a uniform estimate

$$||h||_{C^{m,\alpha}} \leq C(||\hat{L}(h)||_{C^{m-2,\alpha}} + ||B_j(h)||_{C^{m-j,\alpha}} + ||h||_{C^0}), \tag{3.10}$$

where B_j is the part of the boundary operator of order j, together with such an estimate for the adjoint operator. As seen below, the boundary value problem is self-adjoint, so it suffices to establish (3.10).

First, it is simple to prove (3.10) for L in place of $\hat{L}$ via a slight modification of the proof of Proposition 3.1. Namely, for the boundary condition $[h^T]_0 = 0$, we have $h^T = \phi\gamma$ on ∂N, (in place of (3.5)). Note also that (3.3)-(3.4) hold, but without the trh terms. The analog of (3.3) then gives

$$|\xi| h_{0k} = i\xi_k\phi,$$

and hence, via the analog of (3.4),

$$|\xi|^2 h_{00} = -|\xi|^2\phi,$$

so that $h_{00} + \phi = 0$. Next, via the condition $H'_h = 0$, the analog of (3.8) becomes

$$\sum_{k \geq 1}(|\xi| h_{kk} + 2i\xi_k h_{0k}) = 0,$$

which gives

$$n|\xi|\phi = -2i\sum \xi_k h_{0k} = 2|\xi|\phi.$$

Since $n \geq 3$, this implies $\phi = 0$, and so $h_{00} = 0$, hence $h_{0k} = 0$. It follows that $h = 0$, which proves ellipticity of L with the boundary conditions (3.9). Thus, (3.10) holds with L in place of $\hat{L}$.

Next, one has

$$\hat{L} = L - (D^2 trh - \Delta trh\, g) - \delta\delta h\, g. \tag{3.11}$$

Thus to prove (3.10), it suffices to prove

$$||\delta h||_{C^{m-1,\alpha}} \leq C(||\hat{L}(h)||_{C^{m-2,\alpha}} + ||B_j(h)||_{C^{m-j,\alpha}} + ||h||_{C^0}), \tag{3.12}$$

$$||D^2 trh||_{C^{m-2,\alpha}} \leq C(||\hat{L}(h)||_{C^{m-2,\alpha}} + ||B_j(h)||_{C^{m-j,\alpha}} + ||h||_{C^0}). \tag{3.13}$$

From (2.15)-(2.16) and the Bianchi identity, (as in (2.13)), one has $\delta\hat{L}(h) = 2\delta\delta^*(\delta h)$ and the operator $\delta\delta^*$ is elliptic with respect to Dirichlet boundary conditions. Since the boundary data δh in (3.9) is included in the boundary operators B_j, this proves (3.12).

Using this and taking the trace of (3.11) shows that

$$||D^2 trh||_{C^{m-2,\alpha}} \leq C(||\hat{L}(h)||_{C^{m-2,\alpha}} + ||B_j(h)||_{C^{m-j,\alpha}} + ||NN(trh)||_{C^{m-2,\alpha}} + ||h||_{C^0}),$$

so that it suffices to prove that the boundary conditions B cover $NN(trh)$. For this, a simple computation using (3.7), (cf. also (3.19) below), gives

$$N(trh) = 2H'_h - \delta((h(N))^T) - (\delta h)(N) + O(h), \tag{3.14}$$

where $O(h)$ is of order 0 in h. Using the standard interpolation $||h||_{C^{m-1,\alpha}} \leq \varepsilon||h||_{C^{m,\alpha}} + \varepsilon^{-1}||h||_{C^0}$ shows that it suffices here and below only to consider terms with the leading number of derivatives of h.

Now the Gauss equations at ∂N are $|A|^2 - H^2 + s_{\gamma_N} = s_{g_N} - 2Ric(N,N)$ and hence,

$$(|A|^2 - H^2 + s_{\gamma_N})'_h = -2\hat{L}(h)(N,N) + 2\delta^*\delta(h)(N,N) + O(h).$$

One has $s'_{\gamma_N}(h^T) = -\Delta trh^T + \delta\delta(h^T) + O(h^T)$ and A'_h, H'_h only involve first order derivatives in h. Writing then $h^T = B_0(h) + \frac{1}{n}tr_{\partial N}h\gamma_N$, it follows that $tr_{\partial N}h$ at ∂N is controlled by $\hat{L}(h)$, $B_j(h)$, in that

$$||tr_{\partial N}h||_{C^{m,\alpha}} \leq C(||h||_{C^{m-1,\alpha}} + ||\hat{L}(h)||_{C^{m-2,\alpha}} + ||B_j(h)||_{C^{m-j,\alpha}}),$$

and hence

$$||h^T||_{C^{m,\alpha}} \leq C(||h||_{C^{m-1,\alpha}} + ||\hat{L}(h)||_{C^{m-2,\alpha}} + ||B_j(h)||_{C^{m-j,\alpha}}),$$

i.e. h^T is controlled at ∂N by $\hat{L}(h)$ and $B_j(h)$. Next, at ∂N, one has $-(\delta h)(T) = \nabla_N h(N,T) + \nabla_{e_i}h(e_i,T)$, which then gives control as above on $(\nabla_N h)(N,T)$, and so control on $\nabla_N(h(N)^T)$. In turn, this gives then control on $\delta_{\partial N}(\nabla_N(h(N)^T))$, which modulo lower order (curvature) terms, equals $N(\delta(h(N)^T))$. The N-derivative of (3.14) also holds and shows that control of $N(\delta(h(N)^T))$ implies control of $NN(trh)$, so that (3.13) holds, provided $N(H'_h)$ is controlled. But the Riccati equation gives $N(H) = -|A|^2 - Ric(N,N)$; taking the linearization of this in the direction h shows that $N(H'_h)$ is indeed controlled by $\hat{L}(h)$ and the boundary conditions B_j. This completes the proof of ellipticity.

Next, we prove the operator $\hat{L}$ with boundary conditions (3.9) is self-adjoint. To begin, integrating the terms in the expression (2.16) for $\hat{L}$ by parts over N gives

$$\int_N \langle D^*D(h), k\rangle + \int_{\partial N}\langle\nabla_N h, k\rangle = \int_N \langle D^*D(k), h\rangle + \int_{\partial N}\langle\nabla_N k, h\rangle,$$

$$\int_N \delta\delta h trk + \int_{\partial N}(\delta h)(N)trk = \int_N \langle h, D^2(trk)\rangle - \int_{\partial N} h(N, dtrk),$$

and

$$\int_N (\Delta trh)trk - \int_{\partial N} N(trh)trk = \int_N (\Delta trk)trh - \int_{\partial N} N(trk)trh.$$

Here the boundary terms on $S(r)$ all tend to 0 as $r \to \infty$, since the components of h and k are in $C^{m,\alpha}_\delta$ and $\delta > \frac{1}{2}$. It follows that

$$\int_N \langle\hat{L}(h), k\rangle + \int_{\partial N}\langle B(h), k\rangle = \int_N \langle\hat{L}(k), h\rangle + \int_{\partial N}\langle B(k), h\rangle, \tag{3.15}$$

where

$$\langle B(k), h\rangle = \langle\nabla_N k, h\rangle + h(N, dtrk) - (\delta k)(N)trh - trhN(trk). \tag{3.16}$$

Setting $Z(k,h) = \langle B(k), h\rangle - \langle B(h), k\rangle$, we thus need to show that

$$\int_{\partial N} Z(h,k) = 0, \tag{3.17}$$

when h, k satisfy the boundary conditions (3.9).

Thus suppose h and k both satisfy (3.9). A simple calculation shows that $(\delta k)(T) = 0$ is equivalent to

$$(\nabla_N k)(N)^T = \delta_{\partial N}(k^T) - \alpha(k(N)), \tag{3.18}$$

where $\alpha(k(N)) = [A(k(N)) + Hk(N)^T]$, (all taken on ∂N), while $(\delta k)(N) = 0$ is equivalent to

$$N(k_{00}) = \delta_{\partial N}(k(N)^T) + \langle A, k\rangle - k_{00}H. \tag{3.19}$$

The same equations hold for h, and one also has

$$h^T = \phi_h\gamma, \quad \text{and} \quad k^T = \phi_k\gamma. \tag{3.20}$$

We thus need to calculate

$$B(k,h) = \langle \nabla_N k, h\rangle + h(N, dtrk) - trhN(trk),$$

and skew-symmetrize. To begin, write $\langle \nabla_N k, h\rangle = \langle (\nabla_N k)(N), h(N)\rangle + \langle (\nabla_N k)(e_i), h(e_i)\rangle$, so that $\langle \nabla_N k, h\rangle = N(k_{00})h_{00} + \phi_h[N(trk) - N(k_{00})] + 2\langle (\nabla_N k)(N), h(N)^T\rangle$, where we have used the relation $tr_\gamma \nabla_N k = tr_N \nabla_N k - N(k_{00})$. Thus, $B(k,h)$ equals

$$N(k_{00})h_{00} + \phi_h[N(trk) - N(k_{00})] + 2\langle (\nabla_N k)(N), h(N)^T\rangle - N(trk)[trh - h_{00}] + \langle h(N)^T, dtrk\rangle. \tag{3.21}$$

By (3.18) and (3.20),

$$2\langle (\nabla_N k)(N), h(N)^T\rangle = -2\langle d\phi_k, h(N)^T\rangle - 2\alpha(k,h) = -2\phi_k\delta_{\partial N}(h(N)^T) - 2\alpha(k,h) + \omega_1$$

where ω_1 is a divergence term and $\alpha(k,h) = \langle \alpha(k(N)), h(N)^T\rangle$. Similarly, by (3.19) and (3.20),

$$N(k_{00}) = \delta_{\partial N}(k(N)^T) + (\phi_k - k_{00})H,$$

where here and in the following $\delta = \delta_{\partial N}$. Note also that $\langle h(N)^T, dtrk\rangle = trk\delta_{\partial N}(h(N)^T) + \omega_2$, where ω_2 is another divergence term. Since (3.17) involves integration over ∂N, in the following we ignore the divergence terms. Substituting these computations in (3.21) gives

$$\delta(k(N)^T)[h_{00} - \phi_h] + \delta(h(N)^T)[trk - 2\phi_k] - (n-1)\phi_h N(trk) + H(\phi_k - k_{00})(h_{00} - \phi_h) - 2\alpha(h,k).$$

When skew-symmetrizing, the last two terms $H(\phi_k - k_{00})(h_{00} - \phi_h) - 2\alpha(h,k)$ cancel, while the first three terms combine to give

$$-(n-1)[\phi_h\delta(k(N)^T) - \phi_k\delta(h(N)^T)] - (n-1)[N(trk)\phi_h - N(trh)\phi_k],$$

or equivalently, (after dividing by $n-1$),

$$-\phi_h[N(trk) + \delta(k(N)^T)] + \phi_k[N(trh) + \delta(h(N)^T)]. \tag{3.22}$$

On the other hand, by (3.6) or (3.7),

$$\begin{aligned} 2(H')_k &= tr[\nabla_N k - 2\delta^*(k(N)^T) - \delta^*(k_{00}N)] \\ &= N(trk) + 2\delta(k(N)^T) - k_{00}H - N(k_{00}). \end{aligned}$$

Substituting (3.19) gives

$$2(H')_k = N(trk) + \delta(k(N)^T) - H\phi_k,$$

so (3.22) becomes

$$-\phi_h[2(H')_k + H\phi_k] + \phi_k[2(H')_h + H\phi_h] = -2\phi_h(H')_k + 2\phi_k(H')_h.$$

This vanishes exactly when H'_k and H'_h vanish. This completes the proof.

■

The main step in the proof of the manifold theorem, (Theorem 3.6), is the following result.

Theorem 3.3. *Suppose $\pi_1(M, \partial M) = 0$ and $m \geq 3$. Then at any $\widetilde{g} \in \mathbb{E}_S$, the map $\hat{\Phi}$ is a submersion, i.e. the derivative*

$$(D\hat{\Phi})_{\widetilde{g}} : T_{\widetilde{g}}Met_D^{m,\alpha}(N) \to T_{\hat{\Phi}(\widetilde{g})}S_\delta^{m-2,\alpha}(N) \tag{3.23}$$

is surjective and its kernel splits in $T_{\widetilde{g}}Met_D^{m,\alpha}(N)$.

Proof: The operator $\hat{L} = 2D\hat{\Phi}_{\widetilde{g}}$ is elliptic in the interior, and the boundary data in Lemma 3.2 give a self-adjoint elliptic boundary value problem. Let $S_B^{m,\alpha}(N)$ be the space of $C^{m,\alpha}$ symmetric bilinear forms on N satisfying the boundary condition $B(h) = 0$ from Lemma 3.2, i.e.

$$B(h) = \{\delta h, [h^T]_0, (H')_h\} = (0, 0, 0).$$

Clearly, $S_B^{m,\alpha}(N) \subset S_D^{m,\alpha}(N)$, where $S_D^{m,\alpha}(N) = T_{\widetilde{g}}(Met_D^{m,\alpha}(N))$. Throughout the following, we set $\widetilde{g} = g$. The operator $\hat{L}$, mapping

$$S_B^{m,\alpha}(N) \to S_\delta^{m-2,\alpha}(N),$$

$$\hat{L}(h) = f, \quad B(h) = 0 \text{ at } \partial N,$$

is then Fredholm, of Fredholm index 0. On $S_B^{m,\alpha}(N)$, the image $Im(\hat{L})$ is a closed subspace of the range $S^{m-2,\alpha}(N)$, of finite codimension, and with codimension equal to dimension of the kernel K.

If $K = 0$, then $\hat{L}$ maps $S_B^{m,\alpha}(N)$ onto $S_\delta^{m-2,\alpha}(N)$, which proves the result. Thus suppose $K \neq 0$. Then as in (3.15), by the self-adjointness, one has for any $h \in S_B^{m,\alpha}(N)$ and $k \in K$,

$$\int_N \langle \hat{L}(h), k \rangle = \int_N \langle h, \hat{L}(k) \rangle = 0,$$

since the boundary terms vanish and $\hat{L}(k) = 0$. Thus $Im(\hat{L}|_{S_B^{m,\alpha}(N)}) = K^\perp$, (where $K^\perp$ is taken with respect to the L^2 inner product). To prove surjectivity on $S_D^{m,\alpha}(N)$, it thus suffices to prove that for any $k \in K$, there exists $h \in S_D^{m,\alpha}(N)$ such that

$$\int_N \langle \hat{L}(h), k \rangle \neq 0. \tag{3.24}$$

Suppose then (3.24) does not hold, so that

$$\int_N \langle \hat{L}(h), k \rangle = 0, \tag{3.25}$$

for all $h \in S_D^{m,\alpha}(N)$, i.e. for which $\delta h = 0$ on ∂N. Integrating by parts, it follows that

$$\int_N \langle h, \hat{L}(k) \rangle + \int_{\partial N} Z(h, k) = 0, \tag{3.26}$$

for $Z(h, k)$ as following (3.16). As before, the boundary terms at infinity vanish, since $\delta > \frac{1}{2}$.

Choosing $h \in S_D^{m,\alpha}(N)$ arbitrary of compact support in N, it follows from (3.26) that

$$\hat{L}(k) = 0, \tag{3.27}$$

i.e. k is formally tangent to $\hat{Z} = \hat{\Phi}^{-1}(0)$. Of course this is already known, since $k \in K$. Moreover, one also has

$$\delta k = 0 \text{ on } N. \tag{3.28}$$

To see this, let $h = \delta^* V$, with V any vector field vanishing on ∂N. Since g is Einstein and so $(Ric - \frac{s}{2}g)'_{\delta^* V} = 0$, it follows from (3.5) and (3.6) that $\hat{L}(h) = \delta^* Y$, where $Y = 2\delta\delta^* V$. As in Lemma 2.2, the operator $\delta\delta^*$ is surjective, (in fact an isomorphism), on vector fields vanishing at

∂N, so that Y may be arbitrarily prescribed. Moreover, $h \in S^{m,\alpha}_D(N)$ if and only if $Y = 0$ at ∂N. Then (3.25) gives

$$0 = \int_N \langle \hat{L}(\delta^* V), k\rangle = \int_N \langle \delta^* Y, k\rangle = \int_N \langle Y, \delta k\rangle + \int_{\partial N} k(Y, N) = \int_N \langle Y, \delta k\rangle,$$

since $Y = 0$ on ∂N. Here we have used again the fact that the boundary term at infinity vanishes, since $|k| = O(r^{-\delta})$ and $|Y| = O(r^{-1-\delta})$. Since Y is otherwise arbitrary, this gives (3.28).

Returning now to (3.26), (3.27) gives

$$\int_{\partial N} Z(h,k) = 0, \tag{3.29}$$

for all h with $\delta h = 0$ on ∂N. Next, we choose certain test forms $h \in S_D(N)$ in (3.29). First, choose h such that $h = 0$ on ∂N. Then $\nabla_N h$ is freely specifiable, subject to the divergence constraint $\delta h = 0$; all computations here and below are at ∂N. Since $h = 0$, this constraint gives $(\nabla_N h)(N) = 0$, which is equivalent to the tangential and normal constraints:

$$(\nabla_N h)(N, T) = 0, \tag{3.30}$$

$$N(h_{00}) = 0, \tag{3.31}$$

for any T tangent to ∂N. Choosing h and $\nabla_N h$ satisfying $h = 0$ and (3.30)-(3.31) at ∂N, the terms $(\nabla_N h)(T_1, T_2)$ are freely specifiable on ∂N, where T_1, T_2 are any vectors tangent to ∂N. Substituting such h in (3.29) and using (3.28), it follows that

$$\int_{\partial N} \langle \nabla_N h, k\rangle + (k_{00} - trk)N(trh) = 0. \tag{3.32}$$

Now choose $\nabla_N h = fg^T$, where $g^T = g|_{T(\partial N)}$. This choice satisfies the constraints (3.30)-(3.31). The integrand in (3.32) then becomes $ftr^T k - N(trh)tr^T k$. Since $N(trh) = \langle \nabla_N h, g\rangle = nf$, and since f is arbitrary, it follows that $tr^T k = 0$. In turn, since the tangential part of $\nabla_N h$ is arbitrary, (3.32) implies

$$k^T = 0, \quad \text{on} \quad \partial N. \tag{3.33}$$

Lemma 3.4. *At ∂N, one has*

$$(A'_k)^T = 0, \tag{3.34}$$

i.e. $(\nabla_N k)^T = 2[\delta^*(k(N)^T)]^T + k_{00}A$, *since* $k^T = 0$, *cf.* (3.6).

Proof: The proof is a straightforward, but rather long computation. To begin, as preceding (3.21) and using (3.33), one has $\langle \nabla_N h, k\rangle = 2\langle (\nabla_N h)(N), k(N)^T\rangle + N(h_{00})k_{00}$. By (3.18), $(\nabla_N h)(N)^T = \delta_{\partial N}(h^T) - \alpha(h(N))$, so that

$$\begin{aligned} \int_{\partial N} \langle \nabla_N h, k\rangle &= \int_{\partial N} 2\langle \delta_{\partial N}(h^T), k(N)^T\rangle + N(h_{00})k_{00} - 2\alpha(h,k) \\ &= \int_{\partial N} 2\langle h^T, (\delta_{\partial N})^*(k(N)^T)\rangle + N(h_{00})k_{00} - 2\alpha(h,k). \end{aligned} \tag{3.35}$$

Further, for Z tangent to ∂N, one has $(\delta_{\partial N})^*(k(N)^T)(Z,Z) = \langle \nabla^T_Z k(N)^T, Z\rangle = \langle \nabla_Z k(N)^T, Z\rangle = \delta^*(k(N)^T)(Z,Z)$, where now δ^* is taken with respect to the ambient metric g_N, (not the boundary metric γ_N). So this gives

$$\int_{\partial N} \langle \nabla_N h, k\rangle = \int_{\partial N} \langle h^T, 2\delta^*(k(N)^T)\rangle + N(h_{00})k_{00} - 2\alpha(h,k). \tag{3.36}$$

On the other hand, one computes $\langle \nabla_N k, h\rangle = \langle(\nabla_N k)^T, h^T\rangle + \langle \nabla_N k(N), h(N)\rangle = \langle(\nabla_N k)^T, h^T\rangle - \langle \alpha(k(N)), h(N)^T\rangle + N(k_{00})h_{00}$, again by (3.18) and (3.33). Taking the difference of this with (3.36) and noting that α is symmetric, gives

$$\int_{\partial N} \langle h^T, (\nabla_N k)^T - 2\delta^*(k(N)^T)\rangle + N(k_{00})h_{00} - N(h_{00})k_{00} = E, \tag{3.37}$$

where via (3.16)-(3.17), E is given by

$$E = \int_{\partial N} [k(N, dtrh) - h(N, dtrk)] - [N(trh)trk - trhN(trk)].$$

Computing this term-by-term gives: $k_{00}N(trh) + \langle k(N)^T, d^T trh\rangle - h_{00}N(trk) - \langle h(N)^T, d^T trk\rangle - N(trh)trk + trhN(trk)$. Since $trk = k_{00}$, the first and second-to-last terms cancel. Integrating over ∂N and using the divergence theorem shows that

$$E = \int_{\partial N} trh\delta_{\partial N}(k(N)^T) - k_{00}\delta^T(h(N)^T) - h_{00}N(trk) + trhN(trk). \tag{3.38}$$

Next we claim that

$$\delta_{\partial N}(h(N)^T) = N(h_{00}) + Hh_{00} - \langle A, h\rangle, \tag{3.39}$$

and similarly for k. This follows from the following computation: $\delta_{\partial N}(h(N)^T) = \delta_{\partial N}(h(N)) - \delta^T(h_{00}N) = \delta_{\partial N}(h(N)) + Hh_{00}$, while $\delta_{\partial N}(h(N)) = \delta(h(N)) + N(h_{00})$. Since $\delta(h(N)) = (\delta h)(N) - \langle A, h\rangle$, this gives the claim. Substituting (3.39) into (3.38), and using $\langle A, k\rangle = 0$ implies that

$$E = \int_{\partial N} trh(N(k_{00}) + Hk_{00}) - k_{00}(N(h_{00}) + Hh_{00} - \langle A, h\rangle) - h_{00}N(trk) + trhN(trk),$$

and rearranging terms gives

$$E = \int_{\partial N} \langle A, h\rangle k_{00} + N(trk)[trh - h_{00}] + trhN(k_{00}) - k_{00}N(h_{00}) + H(trhk_{00} - trkh_{00}). \tag{3.40}$$

Now substitute (3.40) into (3.37): the $k_{00}N(h_{00})$ term cancels to give

$$\int_{\partial N} \langle h^T, (\nabla_N k)^T - 2\delta^*(k(N))^T\rangle - \langle A, h\rangle k_{00} = \tag{3.41}$$

$$-\int_{\partial N} N(k_{00})h_{00} - N(trk)[trh - h_{00}] - trhN(k_{00}) - H(trhk_{00} - trkh_{00}).$$

The integrand on the right combines to: $-N(k_{00})(h_{00} - trh) - N(trk)[h_{00} - trh] - Htrk(h_{00} - trh) = -[N(k_{00}) + N(trk) + Htrk](h_{00} - trh)$. Since $h_{00} - trh = -tr^T h = -\langle h^T, g^T\rangle$, and since h^T may be chosen arbitrarily, (the constraint $\delta h = 0$ imposes no constraint on h^T), it follows that

$$(\nabla_N k)^T = 2[\delta^*(k(N)^T)]^T + k_{00}A + [N(k_{00}) + N(trk) + Htrk]g^T. \tag{3.42}$$

To complete the proof of (3.34), we thus need to show that

$$N(k_{00}) + N(trk) + Htrk = 0. \tag{3.43}$$

To obtain (3.43), take the g^T-trace of (3.42). One has $\langle \nabla_N k, g^T\rangle = N(trk) - N(k_{00})$, while $\langle \delta^*(k(N)^T), g^T\rangle = \langle \nabla_{e_i} k(N)^T, e_i\rangle = \langle \nabla_{e_i} k(N), e_i\rangle - k_{00}H = \langle (\nabla_{e_i} k)(N), e_i\rangle - k(\nabla_{e_i} N, e_i) - k_{00}H = -N(k_{00}) - k_{00}H$, the last equality using (3.33) and (3.28). This gives

$$N(trk) - N(k_{00}) = -2N(k_{00}) - 2k_{00}H + k_{00}H - n[N(k_{00}) + N(trk) + Htrk],$$

which implies (3.43). This completes the proof of the Lemma.

■

To complete the proof of Theorem 3.3, (3.33) and (3.34) show that

$$k^T = (A'_k)^T = 0,$$

at ∂N. One also has $\hat{L}(k) = \delta k = 0$ on N, so that k is an infinitesimal Einstein deformation on N. By the local unique continuation result of [AH], together with the global hypothesis $\pi_1(M, \partial M) = 0$, it follows that $k = 0$. This shows that $\hat{L}$ is surjective. The fact that its kernel splits is standard, cf. [An3]. This completes the proof.

$\blacksquare$

Via the implicit function theorem, one obtains:

Corollary 3.5. *Suppose $\pi_1(M, \partial M) = 0$ and $m \geq 3$. Then the local spaces $\mathbb{E}_D^{m,\alpha}$ are infinite dimensional C^∞ Banach manifolds, with*

$$T_{\widetilde{g}}\mathbb{E}_D = Ker(D\hat{\Phi}_{\widetilde{g}}). \tag{3.44}$$

Proof: This is an immediate consequence of Theorem 3.3, the fact from Proposition 2.1 that $\mathbb{E}_D = Z_D$, (cf. (2.18)), and the implicit function theorem, (or regular value theorem), in Banach spaces. $\blacksquare$

This leads to the main result of this section.

Theorem 3.6. *Suppose $\pi_1(M, \partial M) = 0$ and $m \geq 3$. Then the moduli space $\mathcal{E}_S = \mathcal{E}_S^{m,\alpha}$ is a C^∞ smooth infinite dimensional Banach manifold for which the boundary map*

$$\Pi_B : \mathcal{E}_S \to Met^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \tag{3.45}$$

is a C^∞ smooth Fredholm map, of Fredholm index 0.

Proof: Recall from §1 that the moduli space $\mathcal{E}_S$ of static vacuum Einstein metrics is defined to be the quotient $\mathbb{E}_S^{m,\alpha}/\mathcal{D}_1^{m+1,\alpha}$. The local spaces $\mathbb{E}_D$ are smooth Banach manifolds and depend smoothly on the background metric $\widetilde{g}$, since the divergence-free gauge condition (2.18) varies smoothly with $\widetilde{g}$. As noted preceding Lemma 2.2, the action of $\mathcal{D}_1$ on $\mathbb{E}$ is free and the local spaces $\mathbb{E}_D$ are smooth local slices for the action of $\mathcal{D}_1$ on $\mathbb{E}_S$. Hence the global space $\mathbb{E}_S$ is a smooth Banach manifold, as is the quotient $\mathcal{E}_S$. The local slices $\mathbb{E}_D$ represent local coordinate patches for $\mathcal{E}_S$.

Proposition 3.1 implies that the boundary map $\Pi_B : \mathbb{E}_S^{m,\alpha} \to Met^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$ is smooth and Fredholm, of Fredholm index 0. Moreover, Π_B is invariant under the action of $\mathcal{D}_1^{m+1,\alpha}(M)$ on $\mathcal{E}_S^{m,\alpha}$ and so it descends to a smooth Fredholm map as in (3.45), still of index 0.

$\blacksquare$

4. Positive mean curvature foliations.

In contrast to the discussion in the previous section which held in all dimensions and for almost arbitrary topologies, we assume here that $dim M = 3$ and further that $\partial M \simeq S^2$ with M diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is a 3-ball.

Recall from §1 that $\mathcal{E}^{nd} \subset \mathcal{E}_S^{m,\alpha}$ is the space of static vacuum solutions (M, g, u) for which the boundary ∂M has positive mean curvature and satisfies the nondegeneracy condition (1.6),

$$|\overline{\nabla} H| \neq 0 \ \text{ on } \ \{K \leq 0\}.$$

Throughout the rest of the paper, the connection on M will be denoted by ∇ and the induced connection on a surface $\Sigma \subset M$ will be denoted by $\overline{\nabla}$.

The purpose of this section is to prove that solutions $(M, g, u) \in \mathcal{E}^{nd}$ have outer-minimizing boundary (1.10), i.e.

$$area(\Sigma) \geq area(\partial M), \tag{4.1}$$

for any surface Σ enclosing ∂M. This is established by proving that all solutions in $\mathcal{E}^{nd}$ admit a foliation Σ_t by surfaces of positive mean curvature with $\Sigma_0 = \partial M$. It is well-known, (via a standard maximum principle argument), that the existence of such a foliation implies (4.1).

To begin, for technical reasons it will be convenient to have a more precise lower bound for the Gauss curvature at critical points of the mean curvature. Let $\varepsilon : \mathcal{E}^{nd} \to \mathbb{R}^+$ be a small error function, tending to 0 sufficiently fast at the boundary of $\mathcal{E}^{nd}$; thus $\varepsilon \leq \varepsilon_0$, where ε_0 is a fixed small constant, and for any $\varepsilon_0 \geq s > 0$, the set of (M, g, u) for which $\varepsilon(M, g, u) \geq s$ is a 'weakly' compact set in $\mathcal{E}^{nd}$, i.e. compact in the $C^{m,\alpha'}$ norm, $\alpha' < \alpha$. Now given a positive constant $w_0 < 1$, define $\mathcal{E}^{nd}_{w_0} \subset \mathcal{E}^{nd}$ to be the space of static vacuum solutions with boundary having positive mean curvature and the following additional properties at ∂M:

$$|\overline{\nabla} \log H|^2 + \varepsilon K > w_0^{-1} - 1 + \theta\varepsilon, \quad \text{and} \tag{4.2}$$
$$1 < w_0^{-1} < 1 + \varepsilon,$$

where $\theta = \frac{1}{2} \min_{\{|\overline{\nabla} H| = 0\}} |K| > 0$. Observe that given any solution $(M, g, u) \in \mathcal{E}^{nd}$, there exists w_0 with $w_0^{-1} - 1$ sufficiently small, such that $(M, g, u) \in \mathcal{E}^{nd}_{w_0}$. Thus $\mathcal{E}^{nd} = \cup_{w_0} \mathcal{E}^{nd}_{w_0}$. It is clear that the spaces $\mathcal{E}^{nd}_{w_0}$ are open submanifolds of $\mathcal{E}_S$.

Next we consider solutions (M, g, u) which admit foliations with leaves of positive mean curvature which are small perturbations of the inverse mean curvature flow (IMCF) starting at ∂M. Such foliations have leaves $\Sigma_t \simeq S^2$ given by level sets of a smooth function $t : M \to \mathbb{R}$, $dt \neq 0$, which in the 0-shift gauge satisfy the evolution equation

$$\partial_t = (wH)^{-1} N. \tag{4.3}$$

Define then $\mathcal{E}^{nd}_{F,w_0} \subset \mathcal{E}^{nd}_{w_0}$ to be the subset of solutions which possess a smooth generalized inverse mean curvature flow (4.3) with $w \in C^m(M)$, which at ∂M satisfy $w|_{\partial M} = w_0$, $\nabla^l_N w|_{\partial M} = 0$ for $2 \leq l \leq m$, $\nabla_N w^{-1}|_{\partial M} = -\varepsilon H^{-1}$ so that $\Delta w^{-1}|_{\partial M} = -\varepsilon$, and with the following additional properties: on all of M

$$1 < w^{-1} < 1 + \varepsilon e^{-t}, \qquad |\nabla^l w^{-1}| < a_l \varepsilon e^{-t}, \qquad \Delta w^{-1} < -\varepsilon^2 e^{-2t} \tag{4.4}$$

where $\{a_l\}_{l=1}^m$ is an appropriately chosen increasing sequence, and on each leaf Σ_t the nondegeneracy condition holds

$$|\overline{\nabla} \log(wH)|^2 + \varepsilon K > w^{-1} - 1 + \theta\varepsilon e^{-t}. \tag{4.5}$$

It is clear that $\mathcal{E}^{nd}_{F,w_0}$ is an open submanifold of $\mathcal{E}^{nd}_{w_0}$. As in §1, we assume here and throughout the following that we are taking the connected component of the standard solution $(\mathbb{R}^3 \setminus B^3(1), g_{flat}, 1)$ in each of the relevant spaces above. The next result proves the statement (4.1).

Theorem 4.1. *The space $\mathcal{E}^{nd}_{F,w_0}$ is closed in $\mathcal{E}^{nd}_{w_0}$, and hence*

$$\mathcal{E}^{nd}_{F,w_0} = \mathcal{E}^{nd}_{w_0}. \tag{4.6}$$

Consequently, every solution $(M, g, u) \in \mathcal{E}^{nd}$ admits a foliation Σ_t of positive mean curvature, with $\Sigma_0 = \partial M$.

Proof: The strategy may be described as follows. Suppose that (M, g_i, u_i) is a sequence of static solutions in $\mathcal{E}^{nd}_{F,w_0}$ which converges to a limit (M, g, u) in $\mathcal{E}^{nd}_{w_0}$ in the $C^{m,\alpha}_\delta$ topology. By assumption each member of the sequence has a foliation $\mathcal{F}_i$ of positive mean curvature, and a function w_i used to construct the foliation. According to the estimates (4.4), the sequence w_i converges (in a subsequence) to a $C^{m-1,1}$ function w. Although the top order derivatives of w are not necessarily continuous, they are appropriately bounded, which is enough for all the arguments to follow; at the end, to regain the full regularity one may use a C^m approximation. Moreover the structure of the foliations $\mathcal{F}_i$ naturally leads to a positive lower and upper bound for the mean curvature in

terms of the ambient geometry. With this estimate and the bounds for w_i, the full geometry of the foliation may be controlled. Thus $\mathcal{F}_i$ will converge (in a subsequence) to a $C^{m-1,\alpha}$ limit foliation $\mathcal{F}$ of (M, g, u). However the limit foliation may have points at which equality holds in (4.4) and (4.5), so the last step entails making a small perturbation to regain the strict inequalities.

We begin by obtaining a uniform lower bound for the mean curvature using an energy method associated to the flow (4.3). The same method gives also a uniform upper bound for H. Let $F(u)$ be an arbitrary smooth positive function of the potential, let $f(t) > 0$ and $\psi(x) \geq 0$ also be arbitrary but smooth, (all three will be chosen below). We set $R_{NN} = Ric(N, N)$. Then by a computation given in the Appendix,

$$f^{-\lambda}\partial_t \int_{\Sigma_t} \psi(x)(f(t)wHF(u))^\lambda = \int_{\Sigma_t} \lambda w\psi F^\lambda (wH)^{\lambda-2} Q, \tag{4.7}$$

where

$$\begin{aligned} Q \;=\; & -(\lambda-1)|\overline{\nabla}\log(wH) + (\log F)' u \overline{\nabla}\log u|^2 - ((\log F)'' + (\log F)'^2)u^2|\overline{\nabla}\log u|^2 \\ & -|A|^2 - (1-(\log F)'u)R_{NN} + \lambda^{-1}H^2 + HN(\log w) + 2H(\log F)'uN(\log u) \\ & + w(\log f)'H^2 - \langle \overline{\nabla}\log(w\psi), (\overline{\nabla}\log(wH) + (\log F)'u\overline{\nabla}\log u)\rangle. \end{aligned} \tag{4.8}$$

We will let $\lambda \to -\infty$, (and later $\lambda \to +\infty$), and will choose the unspecified functions appropriately so that the right side of (4.7) is negative. The choice of the unspecified functions will vary depending on whether the region in question contains critical points of $\log(wH)$. In order to restrict the first variation formula (4.7) to these different types of domains, we will employ ψ as a cut-off function.

To begin consider a region $\Omega \times (s,t)$ on which $|\overline{\nabla}\log(wH)| \neq 0$, and let $\psi \in C_c^\infty(\Omega)$. Then by choosing $F(u) \equiv 1$, all terms in Q will be dominated by $-(\lambda-1)|\overline{\nabla}\log(wH)|^2$ as λ grows large, except perhaps the last term. However the last term may be estimated by

$$\begin{aligned} & |\langle \overline{\nabla}\log(w\psi), (\overline{\nabla}\log(wH) + (\log F)'u\overline{\nabla}\log u)\rangle| \\ \leq\; & \tfrac{1}{2}|\lambda|^{-1}|\overline{\nabla}\log(w\psi)|^2 + \tfrac{1}{2}|\lambda||\overline{\nabla}\log(wH) + (\log F)'u\overline{\nabla}\log u|^2. \end{aligned} \tag{4.9}$$

Therefore, using the term $w(\log f)'H^2$ in (4.8) and by choosing

$$(\log f)' = |\lambda|^{-1}\frac{\int_{\Sigma_t} w\psi(wH)^{\lambda-2}|\overline{\nabla}\log(w\psi)|^2}{\int_{\Sigma_t}\psi(wH)^\lambda}, \tag{4.10}$$

the first variation will be negative.

Now consider a portion of the set $\{|\overline{\nabla}\log(wH)| = 0\}$. The nondegeneracy condition (4.5) implies that we may choose a domain $\Omega \times (s,t)$ containing this portion and on which $K > \frac{1}{2}\theta e^{-t}$. As above we again choose $\psi \in C_c^\infty(\Omega)$ to restrict attention to this domain. Here we can no longer rely on $|\overline{\nabla}\log(wH)|$ to dominate terms in the first variation formula. Thus we must examine the remaining terms more closely. To this end, the Gauss equation on Σ_t gives $|A|^2 - H^2 + 2K = -2R_{NN}$, and hence one has

$$\begin{aligned} -|A|^2 - (1-(\log F)'u)R_{NN} \;=\; & -\tfrac{1}{2}(1+(\log F)'u)|A|^2 + (1-(\log F)'u)K \\ & -\tfrac{1}{2}(1-(\log F)'u)H^2. \end{aligned} \tag{4.11}$$

This along with the coefficient of $|\overline{\nabla}\log u|^2$ in (4.8) suggests that one should choose $F(u)$ to satisfy

$$(\log F)'' + (\log F)'^2 \leq 0, \qquad 1 + (\log F)'u \leq 0. \tag{4.12}$$

Both of these inequalities will hold if $(\log F)' = (2u-c)^{-1}$ (so that $(\log F)'' = -2(\log F)'^2$) with the constant c satisfying $2u < c < 3u$. Notice that it is not necessarily possible to find a constant

having this property globally; however by restricting the size of $\Omega \times (s,t)$ appropriately such a c will exist for the domain in question. The remaining terms in Q form a polynomial in H:

$$\begin{aligned}\left[w(\log f)' - \tfrac{1}{2}(1-(\log F)'u) + \lambda^{-1}\right] H^2 \\ +[N(\log w) + 2(\log F)'uN(\log u)]H \\ +(1-(\log F)'u)K.\end{aligned} \tag{4.13}$$

In light of the lower bound for the Gauss curvature, we may choose

$$(\log f)' = C + |\lambda|^{-1}\frac{\int_{\Sigma_t} w\psi F^\lambda (wH)^{\lambda-2}|\overline{\nabla}\log(w\psi)|^2}{\int_{\Sigma_t}\psi(wHF)^\lambda} \tag{4.14}$$

for some appropriately large constant $C>0$ (depending only on w, u, and the Bartnik data), to guarantee that (4.13) is positive. Moreover the last term in (4.14) is used as before to dominate (4.9). It follows that the first variation formula is again negative.

We have shown how to construct the unspecified functions in domains $\Omega \times (s,t)$, if Ω and $t-s$ are sufficiently small. Therefore

$$\begin{aligned}\int_{\Sigma_t \cap\{\psi=1\}}(f(t)wHF)^\lambda &\le \int_{\Sigma_t}\psi(x)(f(t)wHF)^\lambda \\ &\le \int_{\Sigma_s}\psi(x)(f(s)wHF)^\lambda \\ &\le \int_{\Sigma_s}(f(s)wHF)^\lambda,\end{aligned}$$

and it follows that

$$\left(\int_{\Sigma_s}(f(s)wHF)^\lambda\right)^{1/\lambda} \le \left(\int_{\Sigma_t\cap\{\psi=1\}}(f(t)wHF)^\lambda\right)^{1/\lambda}.$$

Note that the cut-off function ψ is chosen to take the value 1 on a compact set of Ω, and in such a way that (4.10) and the last term in (4.14) converge to zero as $\lambda \to -\infty$. Thus upon taking the limit we obtain

$$\min_{\Sigma_s} f(s)wHF(u) \le \min_{\Sigma_t\cap\{\psi=1\}} f(t)wHF(u), \tag{4.15}$$

where $f(t)$ is either 1 or $\exp(Ct)$.

The desired lower bound for the mean curvature may now be obtained as follows. Cover ∂M by appropriate domains Ω and construct the functions F, f, and ψ as above according to whether or not Ω contains critical points of $\log(wH)$. By applying (4.15) we may then estimate $\min_{\Sigma_t} H$ in terms of $\min_{\partial M} H$ for small t. Next repeat this procedure with ∂M replaced by Σ_t. Hence by working outward we eventually find that for all t

$$\min_{\Sigma_t} H \ge \mathcal{C}(t) > 0, \tag{4.16}$$

where the function $\mathcal{C}(t)$ depends only on the ambient geometry (M,g,u) and the function w. By letting $\lambda \to \infty$, the same arguments give a uniform upper bound $\max_{\Sigma_t} H \le K(t) < \infty$. For this, one changes the sign in (4.12), so that $(\log F)' = -(2u-c)^{-1}$, with $2u < c < 3u$, as well as the signs for $(\log f)'$ in (4.10) and (4.14).

Hence the lapse $(wH)^{-1}$ of a foliation $\mathcal{F}$ as above is uniformly controlled by the ambient geometry (M,g,u) and the given bounds on w. This applies in particular to the sequence of foliations $\mathcal{F}_i$ preceding (4.7).

Moreover, with these bounds for the mean curvature in hand, it is straight forward to obtain a bound for the second fundamental form by following Smoczyk's arguments [Smo], (see also [HI]).

More precisely there exists a constant C depending only on the ambient geometry, the bounds for the mean curvature, $\min w$, $|w|_{C^2}$, and $\max_{\partial M}(wH)^2(1+|A|^2)$ such that $(wH)^2|A|^2 \le C(1+t^2)$. Furthermore by applying the estimates of Krylov [K] in the standard way, we may use the C^0 bound for the second fundamental to obtain control over all higher order derivatives: there exist constants $C(l,\beta,t)$ depending only on the ambient geometry, $\min w$, and $|w|_{C^{l-1,\beta}}$ such that $|A|_{C^{l-1,\beta}} \le C(l,\beta,t)$, $l \le m$. Hence the foliations $\mathcal{F}_i$ converge appropriately (in a subsequence) to a limit foliation $\mathcal{F}$.

As described earlier, the limit foliation $\mathcal{F}$ and function w may admit equalities in (4.4) and (4.5). However here we will show that it is possible to perturb the foliation to regain strict inequalities. Each leaf Σ_t will be deformed in the normal direction by an amount μz_t, where μ is a small parameter. For convenience we will drop the subscript t when there is no danger of confusion. The first task is to compute the new lapse function. Let $v=(wH)^{-1}$ be the old lapse and let tilde signify a quantity after perturbation. Then up to first order the new lapse is given by

$$\begin{aligned} v+\mu\partial_t z &= (w^{-1}+\mu H\partial_t z)H^{-1} \\ &= (w^{-1}+\mu H\partial_t z)(\widetilde{H}^{-1}-\mu DH^{-1}[z])+O(\mu^2) \\ &= (w^{-1}+\mu H\partial_t z-\mu w^{-1}HDH^{-1}[z])\widetilde{H}^{-1}+O(\mu^2). \end{aligned}$$

It follows from the formula for the variation of H, (2^{nd} variation of area), that

$$\begin{aligned} \widetilde{w}^{-1} &= w^{-1}+\mu(H\partial_t z-w^{-1}HDH^{-1}[z])+O(\mu^2) \\ &= w^{-1}+\mu[H\partial_t z-(wH)^{-1}(\overline{\Delta}z+(|A|^2+R_{NN})z)]+O(\mu^2). \end{aligned} \tag{4.17}$$

In order to ensure that (4.4) and (4.5) regain strict inequality after perturbation, we choose z in the following way. Let $\chi\in C^\infty(M)$ satisfy $\chi=O(\varepsilon^2e^{-2t})$ (this function will be specified in (4.21) below), and let $\phi\in C^\infty(M)$ be a cut-off function agreeing with $w^{-1}-\chi$ to all orders at ∂M and vanishing for $t\ge t_0$ where t_0 is small and depends on the ambient geometry. Then define z to be the solution of the Cauchy problem

$$H\partial_t z-(wH)^{-1}(\overline{\Delta}z+(|A|^2+R_{NN})z)=\phi+\chi-w^{-1} \quad \text{on} \quad M, \qquad z|_{\partial M}=0. \tag{4.18}$$

The choice of ϕ implies that z vanishes identically at the boundary, and hence $\widetilde{w}^{-1}$ agrees there with w^{-1} to all orders.

Lemma 4.2. *Equation* (4.18) *admits a unique solution. Furthermore there exists a constant C depending only on the ambient geometry such that*

$$\sup_{\Sigma_t}(|z|+|\overline{\nabla}z|+|\overline{\nabla}^2z|+|\partial_t z|+|\overline{\nabla}\partial_t z|)\le Ce^{t/2}. \tag{4.19}$$

Proof: When written in a more conventional form, equation (4.15) becomes

$$\partial_t z-w^{-1}H^{-2}\overline{\Delta}z-w^{-1}H^{-2}(|A|^2+R_{NN})z=H^{-1}(\phi+\chi-w^{-1}).$$

This is a linear parabolic equation and thus the Cauchy problem admits a unique solution for all time. The bounds (4.19) will follow from the maximum principle and Schauder estimates. First observe that by setting $z=(1+t)^\sigma e^{t/2}\overline{z}$ we obtain

$$\begin{aligned} &\partial_t\overline{z}-w^{-1}H^{-2}\overline{\Delta}\overline{z}-w^{-1}H^{-2}\left(|A|^2+R_{NN}-\left(\tfrac{1}{2}+\sigma(1+t)^{-1}\right)wH^2\right)\overline{z} \\ &= (1+t)^{-\sigma}e^{-t/2}H^{-1}(\phi+\chi-w^{-1}). \end{aligned} \tag{4.20}$$

Under IMCF (or the current small perturbation of IMCF) the mean curvature and second fundamental form fall-off like $e^{-t/2}$, and furthermore the leaves approximate round spheres at infinity.

Therefore if $\sigma > 0$ then the zeroth order coefficient satisfies

$$\begin{aligned}
&|A|^2 + R_{NN} - \left(\tfrac{1}{2} + \sigma(1+t)^{-1}\right) wH^2 \\
&= -\sigma(1+t)^{-1} wH^2 + \left(|A|^2 - \tfrac{1}{2}H^2\right) + \tfrac{1}{2}(1-w)H^2 + R_{NN} \\
&\le -c\sigma(1+t)^{-1}e^{-t} + O(e^{-3t/2}),
\end{aligned}$$

where $c > 0$. Since the right-hand side of (4.20) falls-off like $t^{-\sigma}$, it follows from a standard maximum principle argument that $|\overline{z}(x,t)| \le C(1+t)^{-\sigma}$ where C depends only on the ambient geometry, and hence $|z(x,t)| \le Ce^{t/2}$. Moreover since all the coefficients of (4.20) and their derivatives are uniformly bounded, we may apply the local Schauder estimates on domains $\Omega_T = \partial M \times (T_0 - T, T_0 + T)$ to find

$$\begin{aligned}
&\| \, \overline{z} \, \|_{C^{3+\alpha,3/2+\alpha/2}(\Omega_{T/2})} \\
\le \quad & C_T \left(\| \, (1+t)^{-\sigma} e^{-t/2} H^{-1}(\phi + \chi - w^{-1}) \, \|_{C^{1+\alpha,1/2+\alpha/2}(\Omega_T)} + \| \, \overline{z} \, \|_{C^0(\Omega_T)} \right),
\end{aligned}$$

where

$$\| \, f \, \|_{C^{k+\alpha,k/2+\alpha/2}(\Omega_T)} = \sup_{\Omega_T} \sum_{r+2s\le k} |\overline{\nabla}^r \partial_t^s f| + \max_{r+2s=k} [\overline{\nabla}^r \partial_t^s f]_{\alpha,\alpha/2}$$

and

$$[f]_{\alpha,\beta} = \sup_{T_0-T<t<T_0+T} [f(\cdot,t)]_{\alpha,\partial M} + \sup_{x\in\partial M} [f(x,\cdot)]_{\beta,(T_0-T,T_0+T)}$$

are Hölder norms with exponent α in the space variable and β in the time variable. The desired bounds (4.16) now follow from the Schauder estimates, in light of the C^0 estimate that has already been established.

$\blacksquare$

As mentioned above our choice for z will ensure that, after perturbation, strict inequality holds in (4.4) and (4.5). Choose χ to solve

$$\Delta\chi = -2\varepsilon^2 e^{-2t} \quad \text{on} \quad M, \qquad \chi|_{\partial M} = 0, \qquad \chi \to 0 \quad \text{as} \quad |x| \to \infty, \tag{4.21}$$

and choose t_0 sufficiently small (depending only on the ambient geometry) so that equality cannot occur in (4.4) or (4.5) for $0 \le t < t_0$. It follows that if $\Delta w^{-1} = -\varepsilon^2 e^{-2t}$ then

$$\Delta\widetilde{w}^{-1} = \Delta w^{-1} + \mu(\Delta\chi - \Delta w^{-1}) = -\varepsilon^2 e^{-2t} + \mu(\varepsilon^2 - 2\varepsilon^2)e^{-2t} < -\varepsilon^2 e^{-2t}.$$

Of course outside the set on which $\Delta w^{-1} = -\varepsilon^2 e^{-2t}$ we may simply choose μ sufficiently small in order to obtain the same conclusion. Next observe that it is not possible for $w^{-1} = 1$ since w^{-1} is superharmonic; (in fact this is the reason for including the superharmonic condition). Thus we need only consider the case when $w^{-1} = 1 + \varepsilon e^{-t}$. If this happens then

$$\widetilde{w}^{-1} = w^{-1} + \mu(\chi - w^{-1}) = 1 + \varepsilon e^{-t} - \mu(1 + O(\varepsilon)) < 1 + \varepsilon e^{-t},$$

assuming that ε is small. Lastly consider the derivative estimates in (4.4). Pick local coordinates and let β be a multi-index of order l, so that the components of $\nabla^l w^{-1}$ are $\nabla^\beta w^{-1}$. Let $\delta_\beta = \pm 1$ be such that $|\nabla^\beta w^{-1}| = \delta_\beta \nabla^\beta w^{-1}$, and similarly let $\widetilde{\delta}_\beta = \pm 1$ be such that $|\nabla^\beta \widetilde{w}^{-1}| = \widetilde{\delta}_\beta \nabla^\beta \widetilde{w}^{-1}$. Notice that if

$$\operatorname{sgn}\nabla^\beta w^{-1} \ne \operatorname{sgn}\nabla^\beta \widetilde{w}^{-1} = \operatorname{sgn}[(1-\mu)\nabla^\beta w^{-1} + O(b_l \mu \varepsilon^2 e^{-2t})]$$

then

$$|\nabla^\beta w^{-1}| \le 2b_l \mu \varepsilon^2 e^{-2t}$$

for some constant b_l. Now suppose that $|\nabla^\beta w^{-1}| = \delta a_l \varepsilon e^{-t}$ with $\delta \geq a_l^{-1/2}$. Then we have

$$\begin{aligned}
|\nabla^\beta \widetilde{w}^{-1}| &= \widetilde{\delta}_\beta \nabla^\beta \widetilde{w}^{-1} \\
&= \delta a_l \varepsilon e^{-t} + (\widetilde{\delta}_\beta - \delta_\beta)\nabla^\beta w^{-1} - \mu(\delta a_l \varepsilon e^{-t} + (\widetilde{\delta}_\beta - \delta_\beta)\nabla^\beta w^{-1} + O(b_l \varepsilon^2 e^{-2t})) \\
&= \delta a_l \varepsilon e^{-t} - \mu(\delta a_l \varepsilon e^{-t} + O(b_l \varepsilon^2 e^{-2t})) \\
&< (1 - \mu/2)\delta a_l \varepsilon e^{-t}
\end{aligned}$$

if ε is small and the sequence a_l is chosen appropriately. If $\delta < a_l^{-1/2}$ then the same argument shows that $|\nabla^\beta \widetilde{w}^{-1}| < \delta a_l \varepsilon e^{-t} + O(b_l \mu \varepsilon^2 e^{-2t})$. Thus strict inequality in the derivative estimates of (4.4) is achieved by taking ε small and a_l large. This completes the perturbation argument for (4.4).

We now consider (4.5). In order to calculate the first variation observe that (4.17) and (4.18) imply

$$Dw^{-1}[z] := \lim_{\mu \to 0} \mu^{-1}(\widetilde{w}^{-1} - w^{-1}) = H\partial_t z - w^{-1} H D H^{-1}[z] = \chi - w^{-1},$$

so that

$$D\log(wH)[z] = -(wH)(H^{-1}Dw^{-1}[z] + w^{-1}DH^{-1}[z]) = -wH\partial_t z.$$

Note also that

$$Dt[z] = zH\partial_t t = zH.$$

We then have

$$\begin{aligned}
& D(|\overline{\nabla}\log(wH)|^2 + \varepsilon K - w^{-1} + 1 - \theta\varepsilon e^{-t})[z] \\
= \; & -2A(\overline{\nabla}\log(wH), \overline{\nabla}\log(wH))z - 2\langle \overline{\nabla}\log(wH), \overline{\nabla}(wH\partial_t z)\rangle \\
& + \varepsilon DK[z] + w^{-1} - \chi + \theta\varepsilon z H e^{-t}.
\end{aligned}$$

In the case of equality in (4.5), the estimates for w^{-1}, H, $|A|$, and z show that

$$\begin{aligned}
|A(\overline{\nabla}\log(wH), \overline{\nabla}\log(wH))z| &\leq |A||\overline{\nabla}\log(wH)|^2|z| \\
&\leq C(w^{-1} - 1 + \varepsilon|K| + \theta\varepsilon e^{-t}) \\
&\leq C\varepsilon e^{-t}
\end{aligned}$$

and

$$\begin{aligned}
|\langle \overline{\nabla}\log(wH), \overline{\nabla}(wH\partial_t z)\rangle| &\leq |\overline{\nabla}\log(wH)| wH(|\overline{\nabla}\partial_t z| + |\partial_t z||\overline{\nabla}\log(wH)|) \\
&\leq C(w^{-1} - 1 + \varepsilon|K| + \theta\varepsilon e^{-t})^{1/2} \\
&\leq C\varepsilon^{1/2} e^{-t/2}.
\end{aligned}$$

Moreover the variation of the Gauss curvature is given by

$$DK[z] = \overline{\nabla}_{ij}(zA^{ij}) - \overline{\Delta}(zH) - zHK,$$

and therefore in a similar manner

$$\varepsilon|DK[z]| + \theta\varepsilon|z|He^{-t} \leq C\varepsilon.$$

Lastly since $w^{-1} - \chi > 1/2$ it follows that for small ε the first variation is positive. This completes the perturbation argument for (4.5) and thus completes the proof.

■

Remark 4.3. Theorem 4.1 implies that any static vacuum solution $(M, g, u) \in \mathcal{E}_S$ whose boundary ∂M satisfies the non-degeneracy condition (1.6) is outer-minimizing in M, at least if (M, g, u) is in the component of the standard solution. Note that the non-degeneracy condition is local at ∂M; it depends only on the 1$^{\text{st}}$ order behavior of the geometry at ∂M, while the outer-minimizing

property is global in M. To our knowledge, this criterion for outer-minimizing surfaces is new even for the case of the simplest static vacuum solution, i.e. $\mathbb{R}^3$.

Of course, there are many boundaries which are not outer-minimizing. As a simple illustrative example, let T^2 be a torus of revolution embedded in $\mathbb{R}^3$ with $H > 0$. One may remove a (small) essential annulus from T^2 and smoothly attach two embedded discs to obtain a 2-sphere S^2 with $H > 0$. This construction may be performed to obtain a curve $(S^2)_t$, $t \in [0, 1]$, of positive mean curvature spheres which for $t < \frac{1}{2}$ are embedded and for $t \geq \frac{1}{2}$ are immersed, with a single self-intersection point of the discs at $t = \frac{1}{2}$. For $t < \frac{1}{2}$ but close to $\frac{1}{2}$, the embedded spheres $(S^2)_t$ are not outer-minimizing in $\mathbb{R}^3$, (and don't satisfy the non-degeneracy condition (1.6)).

This passage from embedded to immersed behavior also shows that the full boundary map Π_B on $\mathcal{E}^+$ is not proper. It is easy to see that the embedded spheres $(S^2)_t$ give solutions $(M, g_{flat}, 1)$ which are in the component $\mathcal{E}^+$ containing the standard round solution.

We mention again that it is unknown if there are other components $(\mathcal{E}^{nd})'$ of $\Pi_B^{-1}(\mathcal{B}^{nd})$, (or other components $(\mathcal{B}^{nd})'$ of boundary data in $Met^{m,\alpha}(\partial M) \times C_+^{m-1,\alpha}(\partial M)$ satisfying (1.6)), not containing the standard solution. Theorem 4.1 implies that either all solutions (M, g, u) in such components have a global positive mean curvature foliation close to IMCF, or none do.

Remark 4.4. The proof of Theorem 4.1 shows that the lower bound (4.16) on the mean curvature, as well as the corresponding upper bound, is independent of ε, and depends only on the ambient geometry of (M, g, u). Hence one may let $\varepsilon \to 0$ and it follows that $(M, g, u) \in \mathcal{E}^{nd}$ admits a global solution to the (pure) IMCF (4.3), with $w = 1$. We have not been able to prove this directly, since one does not have a direct analog of the non-degeneracy condition (4.5) for the IMCF itself. Instead, one may view the flow (4.4) as a regularization of the IMCF.

5. Curvature estimates and Properness of Π_B.

In this section, we continue to work on (M, g, u) in place of (N, g_N). Let $inj_{\partial M}$ denote the injectivity radius of the normal exponential map from ∂M in M. The main result of this section is the following.

Theorem 5.1. *For $(M, g, u) \in \mathcal{E}^{nd}$, one has a global pointwise estimate*

$$|R| \leq \Lambda, \tag{5.1}$$

on M, where Λ depends only on bounds for the Bartnik data $(\gamma, H) \in \mathcal{B}^{nd}$, $m \geq 2$. Moreover, at ∂M, one has the bounds

$$|A| \leq \Lambda, \quad inj_{\partial M} \geq \Lambda^{-1}, \tag{5.2}$$

The estimates (5.1), (5.2) *also hold for higher derivatives of R and A, up to order $m - 2$, $m - 1$ respectively.*

Proof: For points in the interior of M, of bounded distance away from ∂M, this follows directly from the apriori interior estimates in [An1] which state

$$|R|(x) \leq \frac{\lambda}{t^2(x)}, \quad |d \log u|(x) \leq \frac{\lambda}{t(x)}, \tag{5.3}$$

where $t(x) = dist(x, \partial M)$, where λ is an absolute constant. Similar (scale-invariant) estimates hold for all higher derivatives of R and $\log u$. So one only needs to consider the behavior near ∂M. At ∂M, the Gauss and Gauss-Codazzi (constraint) equations are given by:

$$|A|^2 - H^2 + s_\gamma = -2R_{NN}, \tag{5.4}$$

$$\delta(A - H\gamma) = -u^{-1} D^2 u(N, \cdot). \tag{5.5}$$

Also, $-R_{NN} = -Ric(N,N) = -u^{-1}NN(u) = u^{-1}(\Delta_{\partial M}u + HN(u))$, so that

$$u(|A|^2 - H^2 + s_\gamma) = 2(\Delta_{\partial M}u + HN(u)). \tag{5.6}$$

From (5.4), a bound on $|R|$ at ∂M gives immediately a bound on $|A|$ at ∂M, given control of (γ, H). Similarly, a bound on $|R|$ on M gives a lower bound on the distance d_{con} to the conjugacy locus of the normal exponential map $exp_{\partial M}$.

Now, again under a bound on $|R|$, the outer-minimizing property (4.1) from Theorem 4.1 implies a lower bound on the distance $\delta_{\partial M}$ to the cut locus of $exp_{\partial M}$. To see this, suppose that $\delta_{\partial M} << 1$ but Λ in (5.1) is bounded, $\Lambda \sim 1$. Then since d_{con} is bounded below, there is a geodesic γ of length $\delta_{\partial M}$ in M meeting ∂M orthogonally at points p_1, p_2. Let T be the boundary of the tubular neighborhood of γ of radius r. Then T intersects ∂M in the boundary of two discs D_1, D_2 of radius approximately r, (for r small). If $\delta_{\partial M} << r$, then $areaT < area(D_1 \cup D_2)$. Further, $T \subset M$ is homologous to $D_1 \cup D_2$ in M. Removing then $D_1 \cup D_2$ from ∂M and attaching T shows that ∂M is not outer-minimizing in M, giving a contradiction; (compare with Remark 4.3).

Thus it suffices to obtain a curvature bound at or arbitrarily near ∂M. The higher derivative estimates may then be obtained by standard elliptic regularity methods. The proof of (5.1) is by a blow-up argument. If the curvature bound in (5.1) is false, then there is a sequence $(M, g_i, u_i, x_i) \in \mathcal{E}^{nd}$ with bounded Bartnik boundary data such that

$$|R_{g_i}|(x_i) \to \infty.$$

Without loss of generality, assume that the curvature of g_i is maximal at x_i. We then rescale the metrics g_i to g_i' so that $|R|$ is bounded, and equals 1 at x_i,

$$|R_{g_i'}|(x_i) = 1, \quad |R_{g_i'}|(y_i) \leq 1, \tag{5.7}$$

for any $y_i \in (M, g_i')$. One may also need to rescale the potential u. For reasons that will be clearer below, choose points $y_i \in M$ such that $dist_{g_i'}(y_i, \partial M) = 1$ and $dist_{g_i'}(y_i, x_i) = 1$, and rescale u_i so

$$u_i(y_i) = 1. \tag{5.8}$$

The sequence (M, g_i', u_i) has uniformly bounded curvature and uniform control of the boundary geometry, (boundary metric, 2^{nd} fundamental form and normal exponential map). By (5.8) and the Harnack inequality, the potential u_i is also uniformly bounded in compact sets. It follows from the convergence theorem in [AT] for manifolds-with-boundary that a subsequence converges weakly, (i.e. in $C^{1,\alpha}$), to a $C^{1,\alpha}$ static limit (X, g, u, x) with boundary $(\partial X, \gamma, u)$. By the normalization in (5.7), the limit (X, g) is complete, (without singularities). Since ∂M is outer-minimizing in (M, g_i), the C^0 convergence to the limit implies that ∂X is outer-minimizing in X: if D is any compact smooth domain in ∂X and D' is a surface in X with $\partial D' = \partial D$, then

$$areaD' \geq areaD. \tag{5.9}$$

One has $\partial X = \mathbb{R}^2$, the boundary metric γ is flat, $H = 0$, so ∂X is a minimal surface in X. One has $u > 0$ in the interior of X, (by the maximum principle), but may have $u = 0$ somewhere or everywhere on ∂X. The bound (5.7) and the static equations imply that u_i is uniformly bounded up to ∂X, within bounded distance to x_i and the limit potential u extends at least $C^{1,\alpha}$ up to ∂X.

We will prove below that the convergence to the limit is smooth, so that in particular

$$|R|(x) = 1, \tag{5.10}$$

where $x = \lim x_i$ and $R = R_X$.

On the blow-up limit (X, g), (5.6) holds and becomes

$$\tfrac{1}{2}u|A|^2 = \Delta_{\partial M}u,$$

on ∂X. This equation holds weakly on ∂X with $u \in C^{1,\alpha}(\partial X)$; elliptic regularity then implies it holds strongly, and $u \in C^{3,\alpha}(\partial X)$. Since u is harmonic, u is thus $C^{3,\alpha}$ up to ∂X. Also, by the Riccati equation $N(H) = -|A|^2 - R_{NN} = -|A|^2 + \frac{1}{2}(|A|^2 - H^2 + s_\gamma)$, so that

$$N(H) = -\tfrac{1}{2}(|A|^2 + H^2 - s_\gamma). \tag{5.11}$$

This holds pointwise on the blow-up sequence (M, g_i', u_i) and since $s_\gamma \to 0$ and $H \to 0$ for g_i', it follows that $N(H)$ is defined pointwise on the limit ∂X and on ∂X,

$$N(H) \leq 0, \tag{5.12}$$

with equality on any domain only when $A = 0$.

Since ∂X is minimal, (5.12) and the outer-minimizing property (5.9) imply that

$$N(H) = 0, \tag{5.13}$$

on ∂X. In more detail, (5.9) and the fact that $H = 0$ on ∂X implies the $2^{\rm nd}$ order stability of ∂X, in that the $2^{\rm nd}$ variation of the area of ∂X is non-negative. Thus, for all f of compact support on ∂X, one has

$$\int_{\partial X} (|df|^2 + f^2 N(H)) \geq 0. \tag{5.14}$$

Choose $f = f_{R,S}(r)$ such that $f = 1$ on $D(R) \subset \partial X = \mathbb{R}^2$ and, for $r \geq R$, $f = (\log r - \log S)/(\log R - \log S)$, for $S >> R >> 1$. One may choose R and S sufficiently large such that $\int_{\partial X} |df|^2 < \varepsilon$, for any given $\varepsilon > 0$. This together with (5.12) implies (5.13).

It follows that $A = 0$ and hence by the Liouville theorem on $\mathbb{R}^2$, $u = const$ on ∂X. Using the divergence constraint (5.5), we also now have $0 = \delta(A - H\gamma) = -u^{-1}D^2(N, \cdot)$, and so $0 = D^2(N, \cdot) = dN(u) - A(du) = dN(u)$, so that $N(u) = const$.

Thus, the full Cauchy data $(\gamma, u, A, N(u))$ for the static vacuum equations is fixed and trivial: γ is the flat metric, $A = 0$ and u, $N(u)$ are constant. Observe that this data is realized by the family of flat metrics on $(\mathbb{R}^3)^+$ with either $u = const$ or u equal to an affine function on $(\mathbb{R}^3)^+$.

Suppose $u = const > 0$ on ∂X. The static vacuum equations (1.1) are then non-degenerate up to ∂X. The unique continuation property for Einstein metrics with boundary, cf. [AH], implies that the Cauchy data uniquely determine the solution locally. Alternately, since the static vacuum equations are non-degenerate up to ∂X and since the boundary data $(\partial X, \gamma, H)$ are real-analytic, elliptic regularity implies that the solution (M, g, u) is real-analytic up to ∂M. Such solutions are uniquely determined (locally) by their Cauchy data. Hence, the limit (X, g, u) is flat in this case.

Moreover, the convergence to the limit is smooth everywhere. This again follows from non-degeneracy and ellipticity. Briefly, the potential equation $\Delta u = 0$ gives a boost on the regularity of u, (given background regularity on g). One substitutes this into the main static vacuum equation $uRic = D^2 u$, giving thus a boost to the regularity of Ric, inducing then a boost to the regularity of g. This in turn further boosts the regularity of u via the potential equation. Bootstrapping gives $C^{m,\alpha}$ convergence, up to the boundary, in regions where $u > 0$, (given that u is $C^{m,\alpha}$ at the boundary). In sum, one has a contradiction to (5.10).

Thus, suppose instead

$$u = 0 \ \text{ on } \ \partial X. \tag{5.15}$$

This situation is more complicated. It is also more difficult to prove smooth convergence in this situation (one may have x in (5.10) at ∂X). Moreover, there are in fact non-flat static vacuum solutions with flat Cauchy data as above with $u = 0$ on ∂X, (so-called toroidal black holes, cf. [P] and [T]). Thus the unique continuation results used above are false in this degenerate situation where the boundary becomes characteristic.

We observe first that the solution (X,g) is still real-analytic up to ∂X in this case. This follows since the 4-metric $g^4 = u^2 d\theta^2 + g_X$ is Einstein ($Ric = 0$) and is $C^{1,\alpha}$ up to the horizon or vanishing locus $\partial X = \{u = 0\}$. Elliptic regularity for the Einstein equations then implies that g^4 is real-analytic, and hence so are u, g_X up to ∂X.

To see this in more detail, let U be a chart neighborhood of ∂X in X, so that U is diffeomorphic to a half-ball in $\mathbb{R}^3$ with boundary a disc $D^2 \subset \mathbb{R}^2$. Over each $p \in U$, one has a circle of length $2\pi u(p)$, with $u \to 0$ as $p \to \partial X$. From the work above, u is $C^{1,\alpha}$ up to ∂X with $N(u) = const$ at ∂X. Note that $N(u) \neq 0$ at ∂X. For if $N(u) = 0$ at ∂X, since also $u = 0$ at ∂X and u is harmonic ($\Delta u = 0$), the unique continuation property for harmonic functions implies that $u = 0$ in X, giving a contradiction. By rescaling u if necessary, one may thus assume that $N(u) = 1$ at ∂X. This implies that the 4-metric g^4, defined on $B^4 \setminus D^2$ extends to a $C^{1,\alpha}$ metric on the 4-ball B^4. (The coordinate θ is an angular variable in $\mathbb{R}^2$ in polar coordinates, shrinking down to the origin on approach to ∂X). It is well-known, cf. [Be], that any $C^{1,\alpha}$ weak solution to the Einstein equations is real-analytic (in harmonic or geodesic normal coordinates), which gives the claim above.

To prove the limit is in fact flat, and that one has strong convergence, we need to use the outer-minimizing property again. Thus, first note that (5.11) holds everywhere on the limit (X,g) near ∂X, not just at ∂X; here A is the 2^{nd} fundamental form of the level sets $S(t)$ of $t = dist(\partial X, \cdot)$, etc. We have already established $N(H) = 0$ at ∂X, via the outer-minimizing property and the corresponding stability of the 2^{nd} variation operator (5.14). Taking then the derivative of (5.11) in the normal direction gives,

$$NN(H) = -\langle A', A\rangle - \langle A^2, A\rangle - H'H + \tfrac{1}{2}s'_\gamma, \tag{5.16}$$

where $A' = \nabla_N A$. At ∂X, the first three terms vanish while $(s'_\gamma)_k = -\Delta(trk) + \delta\delta k - \langle Ric_\gamma, k\rangle = 0$, since $k = 2A = 0$. Thus

$$NN(H) = 0, \tag{5.17}$$

at ∂X and it follows that the 3^{rd} variation of the area of ∂X in the unit normal direction vanishes.

Now choose $f = f_{R,S}$ as following (5.14) with R, S large. Let $S_{tf} = exp_{\partial X}(tf(x))$, where $exp_{\partial X}$ is the normal exponential map of ∂X into X. Letting $v(t) = area S_{tf}$, one has

$$v(t) = v(0) + \tfrac{1}{2}v''(0)t^2 + \tfrac{1}{6}v'''(0)t^3 + \tfrac{1}{24}v''''(0)t^4 + O(t^5). \tag{5.18}$$

The expansion (5.18) is valid for all t sufficiently small, $|t| \leq \delta_0$, with δ_0 independent of R, S, since the area and its derivatives are integrals of local expressions, and the local geometry of X is uniformly bounded in a tubular neighborhood of radius 1 about ∂X. By the 2^{nd} variational formula (5.14) and (5.13), for any given $\varepsilon > 0$, one has

$$v''(0) \leq \varepsilon,$$

for R, S sufficiently large. For the same reasons via (5.17),

$$v'''(0) \leq \varepsilon.$$

It follows then from the outer-minimizing property (5.9) and (5.18) that for R, S sufficiently large, one must have

$$v''''(0) \geq -\varepsilon, \tag{5.19}$$

again for any $\varepsilon = \varepsilon(R,S) > 0$. Using the vanishing of the lower order terms, one computes that (5.19) gives

$$\int_{\partial X} f^4 NNN(H) - 6f^2\langle df \cdot df, A'\rangle \geq -\varepsilon. \tag{5.20}$$

On the other hand, taking the normal derivative of (5.16) gives

$$NNN(H) = -\langle A', A'\rangle - (H')^2 + \tfrac{1}{2}s''_{\gamma}. \tag{5.21}$$

We have $H' = N(H) = 0$ and $(s'_{\gamma})_{2A} = 2\Delta H + 2\delta\delta A - \langle Ric_{\gamma}, 2A\rangle$. For s''_{γ}, one has $(\Delta H)' = \Delta' H + \Delta H' = 0$ and $\langle Ric_{\gamma}, A\rangle' = \langle (Ric_{\gamma})', A\rangle + \langle (Ric_{\gamma}), A'\rangle = 0$. So at ∂X, $\frac{1}{2}s''_{\gamma} = \delta\delta A'$. It follows then from (5.20)-(5.21) that for $f = f_{R,S}$ as above

$$\int_{\partial X} -f^4|A'|^2 + f^4\delta\delta A' - 6f^2\langle df\cdot df, A'\rangle \geq -\varepsilon. \tag{5.22}$$

Integrating the second term by parts gives $\int\langle D^2 f^4, A'\rangle = \int\langle 4f^3 D^2 f + 12f^2\langle df\cdot df, A'\rangle$. Using the Cauchy-Schwarz and Young inequalities, (5.22) then implies, for any μ small,

$$\int_{\partial X} f^4|A'|^2 \leq \mu\int_{\partial X} f^4|A'|^2 + C\mu^{-1}\int_{\partial X} f^2|D^2 f|^2 + C\mu^{-1}\int_{\partial X}|df|^4 + \varepsilon.$$

Choosing μ small, the first term on the right may be absorbed into the left, while simple computation shows that the last two terms become arbitrarily small for R and S sufficiently large. It follows that

$$A' = 0 \tag{5.23}$$

and so $NNN(H) = 0$ at ∂X. The Riccati equation

$$A' = \nabla_N A = -A^2 - R_N, \tag{5.24}$$

where $R_N(V,W) = \langle R_g(V,N)N, W\rangle$, thus gives $R_N = 0$ at ∂X, and so via the Gauss and Gauss-Codazzi equations $R_g = 0$ at ∂X. Thus the full ambient curvature vanishes at ∂X.

One can now continue inductively in the same way to see that A and R vanish to infinite order at ∂X. A simpler method proceeds as follows. The Riccati equation (5.24) holds along the level sets $S(t)$ of $t = dist(\partial X, \cdot)$. Since $s_g = 0$ and $dim X = 3$, $R_N = - * Ric$, i.e. $R_N(v,v) = -Ric(w,w)$, where (N, v, w) are an orthonormal basis. Via the static vacuum equations, this gives $\nabla_N A = -A^2 + u^{-1} * (D^2 u)$. Rescale u if necessary so that $N(u) = 1$ at ∂X and set $v = u - t$. Since $A = D^2 t$, one then obtains

$$\nabla_N A = -A^2 + u^{-1}(*A) + u^{-1}(*D^2 v). \tag{5.25}$$

This is a system of ODE's for A, singular at $\partial X = \mathbb{R}^2$, but with indicial root 1. From the work above, we have $v = O(t^2)$ and $A = O(t^2)$. Writing $A = t^2 B$ and substituting in (5.25) shows that $A = O(t^3)$. Also, by the computation following (5.21), $s_{\gamma} = O(t^3)$ on $S(t)$, and hence using the scalar constraint (5.4) and the relation $R_{NN} = u^{-1}NN(u)$, this in turn implies $v = O(t^3)$, and so on. It follows that (g, u) agree with a flat solution to infinite order at ∂X. Since the solution (X, g, u) is analytic up to ∂X, it follows that (X, g, u) is flat, as claimed.

Next we claim that one has strong convergence to the limit, so that (5.7) is preserved in the limit, i.e. (5.10) holds, contradicting the fact that the limit is flat. Note first that if x in (5.10) is in the interior of X, then strong (C^{∞}) convergence is immediate, by the interior estimates (5.3), i.e. their higher derivative analogs. Thus we may assume that $x \in \partial X$.

From the work above, we know that $|R|$ is uniformly bounded everywhere on (M, g'_i) and $|R| \to 0$ everywhere away from $\partial M \to \partial X$, so $|R|$ jumps quickly from 1 to 0 near x_i. The main point is to prove that

$$|R|(x) \to 0, \tag{5.26}$$

for all $x \in \partial M \to \partial X$; it is then easy to prove that $|R| \to 0$ on (M, g'_i), cf. (5.35) below. To prove (5.26), note that the estimates (5.11)-(5.23) above at ∂X also hold on the blow-up sequence at ∂M,

(since $H \to 0$ and $s_\gamma \to 0$ on ∂M). It follows then by these arguments that for any $R < \infty$ and $D(R)$ the R-ball about any base point $y_i \in (\partial M, \gamma_i')$ converging to $y \in \partial X$,

$$\int_{D(R)} |A|^2 + |\nabla_N A|^2 \to 0. \tag{5.27}$$

To proceed further, consider the divergence constraint $\delta A = -dH - Ric(N, \cdot)$ on $(\partial M, \gamma_i')$, as in (5.5). The equations $\delta A = \chi_1$, $dtrA = \chi_2$ form an elliptic first order system in 2-dimensions, and so one has elliptic estimates. Since $H \to 0 \in C^{m-1,\alpha}$ on $(\partial M, \gamma_i')$, $dH \to 0$ in $C^{m-2,\alpha}$. Also, by assumption (5.7), $Ric(N, \cdot)$ is bounded in L^∞. It follows then from elliptic regularity that A is bounded in $L^{1,p}$. By the scalar constraint (5.4), R_{NN} is then also bounded in $L^{1,p}$ and since $trR_N = R_{NN}$, it follows from (5.27) and (5.24) that

$$A \to 0 \ \text{ and } \ R_{NN} \to 0 \ \text{ in } \ C^\alpha_{loc}(\partial M). \tag{5.28}$$

Next the Einstein equation $Ric_{g_N} = 0$ on (N, g_N) implies that $\delta_N R_{g_N} = 0$. Hence

$$0 = (\delta_{g_N} R)((\cdot, N)N, \cdot) = \delta_{g_N}(R(\cdot, N)N, \cdot) + 2R(e_\alpha, \nabla_{e_\alpha} N)N = \delta_{g_N}(R(\cdot, N)N, \cdot),$$

since one may choose a basis in which $\nabla_{e_\alpha} N = A(e_\alpha) = \lambda_\alpha e_\alpha$. Let V be the unit vertical vector, and note that $\nabla_V V = -d\nu$, where $\nu = \log u$. Then $\nabla_V(R(V, N)N, \cdot) = R(d\nu, N)N$, while $\nabla_N(R(N, N)N, \cdot) = 0$. Hence, for R_N as in (5.24), these computations on ∂M give

$$\delta R_N = -R_N(d\nu),$$

where the divergence δ and R_N are taken on $(\partial M, \gamma_i')$. Since R is bounded in L^∞ and $trR_N = R_{NN}$ is bounded in $L^{1,p}$, elliptic regularity gives

$$||R_N||_{L^{1,p}} \leq C||d\nu||_{L^p}, \tag{5.29}$$

again on compact domains in ∂M converging to a compact domain in ∂X.

To control $d\nu$ in (5.29), recall that by (5.7) the ambient curvature R is bounded, and so R restricted to ∂M is also bounded. Via the static vacuum equations (1.1), this implies that

$$N(\nu)A + u^{-1}D^2 u \tag{5.30}$$

is bounded on $(\partial M, \gamma_i')$, where $D^2 u$ is the Hessian of $u|_{\partial M} : \partial M \to \mathbb{R}^+$. Now we claim that each term in (5.30) is bounded, i.e. there exists K such that

$$|u^{-1}D^2 u| \leq K, \ \ |N(\nu)A| \leq K, \tag{5.31}$$

pointwise, on domains converging to a bounded domain in ∂X. To prove (5.31), suppose instead that $|u^{-1}D^2 u| \to \infty$ at some sequence of base points $y_i \to y \in \partial X$. Without loss of generality, we may assume the points y_i realize the maximum of $|u^{-1}D^2 u|$ on $D_{y_i}(10) \subset (\partial M, \gamma_i')$, (possibly up to a factor of 2). As before, one may then rescale the metrics g_i' further to g_i'' so that $|u^{-1}D^2 u|(y_i) = 1$ and hence the full curvature $R \to 0$ in this scale. Also, renormalize u if necessary so that $u(y_i) = 1$. Note that $|du|(y_i)$ must be bounded. For if $|du|(y_i)$ were too large, it follows, (e.g. by a still further rescaling), that u would be close to an affine function on $\mathbb{R}^2$ and hence u would assume negative values in bounded distance to y_i. Since $u > 0$ everywhere, this is impossible. Thus, by integration along paths, u is bounded in $L^{1,\infty}$ in the scale g_i'', within bounded distance to y_i.

Now working in the scale and normalization above, from divergence constraint (5.5), one has $-u\delta(A - H\gamma) = dN(u) - A(du)$. Since A is bounded in $L^{1,p}$ and u and du are bounded in L^∞, it follows that $dN(u)$ is bounded in L^p. Thus, $N(u) = c + \phi$, where ϕ is bounded in $L^{1,p}$. Here c is a constant which may, and in fact does, go to ∞, in the u-normalization $u(y_i) = 1$ above.

The trace equation (5.6) in this scale and normalization gives

$$\Delta u + H(c + \phi) = uf, \tag{5.32}$$

where f is bounded in $L^{1,p}$. Also, Hc is bounded, since $N(u)A$ is bounded via (5.30), and so $Hc \to c'$, for some constant c' on ∂X. Since ϕ is also bounded in $L^{1,p}$, it follows that u is bounded in $L^{3,p}$, and hence (in a subsequence), u converges in $C^{2,\alpha}$ to its limit on $\partial X = \mathbb{R}^2$. Hence D^2u converges in C^α to its limit on $\mathbb{R}^2$. On the limit, since $H \to 0$, the trace equation (5.32) becomes

$$\Delta u + c' = 0. \tag{5.33}$$

If $c' = 0$ then since $u > 0$, $u = const$ and hence $D^2u = 0$, giving a contradiction. If $c' \neq 0$, then again since $u > 0$, one must have $c' < 0$ and u is a quadratic polynomial on $\mathbb{R}^2$. Since u is harmonic on $(\mathbb{R}^3)^+$ and $c' < 0$ implies $N(u) = const < 0$, this is also easily seen to be inconsistent with the requirement $u > 0$ everywhere. This proves (5.31) holds.

Returning to (5.29), we may work in the normalization above that $u(y_i) = 1$ and then (5.31) implies that $d\nu = u^{-1}du$ is bounded in L^∞ in bounded domains about y_i. Hence by (5.29), R_N is bounded in $L^{1,p}$ and so bounded in C^α. Since $R_N \to 0$ in L^2 locally on ∂M, one has

$$R_N \to 0 \ \text{ in } \ C^\alpha_{loc}(\partial M). \tag{5.34}$$

This is the main part of the estimate (5.26).

Next, one needs the same result for the $Ric(N,T)$ term. To do this, take the normal derivative of the scalar constraint (5.6), to obtain

$$\Delta N(u) + \Delta' u + N(H)N(u) + HNN(u) = \tfrac{1}{2}N(u)[|A|^2 - H^2 + s_\gamma] + \tfrac{1}{2}uN[|A|^2 - H^2 + s_\gamma].$$

Since $\gamma' = 2A$, a standard formula for the variation of the Laplacian, (cf. [Be]), gives $\frac{1}{2}\Delta' u = -\langle D^2u, A\rangle + \langle du, \beta(A)\rangle$ which is bounded in L^∞. Also the terms $N(H)$, H and $NN(u) = -\frac{u}{2}[|A|^2 - H^2 + s_\gamma]$ all go to 0 in L^∞. Moreover, via (5.34) above, $uN[|A|^2 - H^2 + s_\gamma]$ is bounded in L^∞. Also, as in the proof of (5.31), $N(u) = c + \phi$ with ϕ bounded in $L^{1,p}$. Thus it follows from the above by elliptic regularity that $dN(u)$ is bounded in $L^{2,p}$ and so converges in $C^{1,\alpha}$ on ∂M. Since

$$-uRic(N,\cdot) = dN(u) - A(du),$$

it now follows that $Ric(N,\cdot)$ converges to its limit, necessarily 0, in C^α on ∂M. Lastly, $\delta A = -dH - Ric(N,\cdot) \to 0$ in C^α and hence by elliptic regularity, $A \to 0$ in $C^{1,\alpha}$ so that via the Gauss-Codazzi equations $dA(X,Y,Z) = \langle R(N,X)Y, Z\rangle \to 0$ in C^α; here d is the exterior derivative. Combining the computations above now proves the estimate (5.26).

To complete the proof, we have $|R| \to 0$ on $\partial M \to \partial X$. On the 4-manifold $N = M \times_u S^1$, the Einstein equations give the inequality

$$\Delta_N |R_{g_N}| + c|R_{g_N}|^2 \geq 0, \tag{5.35}$$

where c is a fixed numerical constant, Δ_N is the 4-Laplacian and R_{g_N} is the curvature tensor on N. One has $|R_{g_N}| = |R|$, (up to a constant). We have proved above that $R \to 0$ in L^2 locally on $(N, (g_N)_i)$ and $R \to 0$ pointwise at ∂N. It follows from the deGiorgi-Nash-Moser estimates for domains with boundary, (cf. [GT, Thm. 8.25]), that $R \to 0$ pointwise on N and hence on M, contradicting (5.7). This completes the proof.

∎

Next, we show that the potential function u is also controlled by the boundary data of Π_B.

Corollary 5.2. *For $(M, g, u) \in \mathcal{E}^{nd}$, there is a constant U_0 depending only on the Bartnik data $(\gamma, H) \in \mathcal{B}^{nd}$ such that*

$$u \leq U_0, \tag{5.36}$$

on M. Moreover, if $H \geq H_0 > 0$ on ∂M, then there exists U_0 as above depending in addition only on H_0, such that on M,

$$u \geq U_0^{-1}. \tag{5.37}$$

Proof: Let $S(s) = \{x \in (M, g) : dist(x, \partial M) = s\}$ be the geodesic 'sphere' about ∂M. Choose a fixed base point $x_0 \in S(1)$ and suppose one has the bound

$$c_0^{-1} \leq u(x_0) \leq c_0. \tag{5.38}$$

By Theorem 5.1, the geometry of the annular region $A(\frac{1}{2}, 2)$ about $S(1)$ is uniformly controlled by the Bartnik data (γ, H) and so by integration of the static vacuum equations $uRic = D^2u$ along paths in $A(\frac{1}{2}, 2)$, one has

$$C_0^{-1} \leq u \leq C_0, \tag{5.39}$$

in $A(\frac{3}{4}, \frac{3}{2})$, where C_0 depends only on c_0 and (γ, H).

To remove the dependence of C_0 in (5.39) on c_0 in (5.38), we need better control on the large-scale behavior of u. To do this, it is proved in [An2, Lemma 3.6], that there is a constant K, depending only on C_0, such that for all $s \geq 1$,

$$\sup_{S_c(s)} |du| \leq K(v_c(s))^{-1}, \tag{5.40}$$

where $v_c(s) = areaS_c(s)$ and $S_c(s)$ is any component of the geodesic sphere $S(s)$. We point out that (5.40) holds for general static vacuum solutions, not only those in $\mathcal{E}^{nd}$ for instance. The estimate (5.40) is proved by studying the behavior of the harmonic potential $\log u$ on the Ricci-flat 4-manifold $(N, g_N, \log u)$ and then reducing to (M, g_M, u).

Consider now the conformally equivalent metric

$$\widetilde{g} = u^2 g. \tag{5.41}$$

It is well-known that the static vacuum Einstein equations (1.1) are equivalent to the equations $\widetilde{Ric} = 2(d\nu)^2 \geq 0$, $\Delta_{\widetilde{g}}\nu = 0$, $\nu = \log u$. The metric $\widetilde{g}$ thus has non-negative Ricci curvature with harmonic potential ν. These are exactly the properties used to prove (5.40), and a brief examination of its proof shows that (5.40) also holds with respect to $\widetilde{g}$, i.e.

$$\sup_{\widetilde{S}_c(s)} |d\nu|_{\widetilde{g}} \leq K(\widetilde{v}_c(s))^{-1}, \tag{5.42}$$

again with $K = K(C_0)$. Since (M, g) is asymptotically flat and $u \to const$ at infinity, the area growth of geodesic spheres $\widetilde{v}(s)$ in $(M, \widetilde{g})$ satisfies $\widetilde{v}(s)/s^2 \to \omega_2$, where $\omega_2 = areaS^2(1)$. It follows then from the volume comparison theorem for Ricci curvature, (cf. [Pe]), that

$$\widetilde{v}(s) \geq \omega_2 s^2, \tag{5.43}$$

for all $s \geq 1$. As above, by integration of (5.42) along a geodesic ray starting from a suitable base point $x_1 \in S(1)$ out to infinity, one sees that (5.36) holds then globally on $M \setminus B(1)$, with U_0 again depending only on c_0 in (5.38). Using the static vacuum equations, the same integration along paths gives such a bound within $B(1)$. Thus, we see that (5.36) follows from (5.38).

To prove (5.38), suppose one has a static vacuum solution (M, g, u) with

$$u(x_0) = \varepsilon. \tag{5.44}$$

Renormalize u to $\bar{u} = u/u(x_0)$, so that $\bar{u}(x_0) = 1$ and $\bar{u} \to \varepsilon^{-1}$ at infinity. Then (5.38) holds, and hence so does (5.42)-(5.43). Again by integration along geodesics starting at x_0 and diverging to infinity, it follows that

$$u \leq U_1,$$

where U_1 depends only on the boundary data of Π_B. This proves the lower bound in (5.38).

To prove the upper bound, suppose instead

$$u(x_0) = \varepsilon^{-1}. \tag{5.45}$$

Then again we renormalize u to $\bar{u}$ as above, so that now $u \to \varepsilon$ at infinity. This does not directly give a lower bound on ε via (5.42)-(5.43) as above. However, one may proceed as follows. First, it is well-known that static vacuum solutions come in "dual" pairs, in that if $(M, g, \bar{u})$ is a static vacuum solution, then so is $(M, \hat{g}, \hat{u})$ with $\hat{g} = \bar{u}^4 g$, $\hat{u} = \bar{u}^{-1}$, cf. [An2] for instance. Then (5.42)-(5.43) hold for $(M, \hat{g}, \hat{u})$ which as before by integration gives an upper bound $\hat{u} \leq U_1$ at infinity. Since near infinity, $\hat{u} \simeq \varepsilon^{-1}$, this again gives a bound on ε^{-1}. This completes the proof of (5.36).

To prove (5.37), note that (5.38) has been proved above, and hence by the maximum principle and normalization $u \to 1$ at infinity, (5.37) holds in the exterior region $M \setminus B(1)$. Thus, one only needs to consider the behavior near ∂M. For this, suppose (M, g, u) is a static vacuum solution, $C^{2,\alpha}$ up to ∂M with $u \geq 0$ on $\bar{M} = M \cup \partial M$. If $H \geq H_0 > 0$ on ∂M, we claim that necessarily

$$u > 0 \;\text{ on }\; \partial M.$$

For if $u = 0$ at some point $z \in \partial M$, then by (5.6), $\Delta_{\partial M} u + HN(u) = 0$ at z. Since $0 = u(z)$ is a global minimum for u, one has $\Delta_{\partial M} u \geq 0$ and by the Hopf maxmimum principle, $N(u) > 0$ at z. This gives a contradiction if $H > 0$. The same arguments prove the existence of a lower bound (5.37) by a contradiction argument, taking a sequence and passing to a limit, using Theorem 5.1.

■

The previous results now lead quite easily to the following main result of this section.

Corollary 5.3. *The boundary map*

$$\Pi_B : \mathcal{E}^{nd} \to \mathcal{B}^{nd},$$

is proper.

Proof: Let (M, g_i, u_i) be a sequence of static vacuum solutions in $\mathcal{E}^{nd}$, with $\Pi_B(g_i, u_i) = (\gamma_i, H_i)$. Supposing $(\gamma_i, H_i) \to (\gamma, H)$ in $\mathcal{B}^{nd}$, we need to prove that the sequence (g_i, u_i) has a subsequence converging in $C^{m,\alpha}(M)$, modulo diffeomorphisms, to a limit $(M, g, u) \in \mathcal{E}^{nd}$.

The curvature bound (5.1) and control of the intrinsic and extrinsic geometries of the boundary metrics first implies the metrics g_i cannot collapse within bounded distance to ∂M, i.e. there is a fixed constant $i_0 > 0$ such that the injectivity radius of (M, g_i) satisfies

$$inj_{g_i}(x) \geq i_0,$$

for $dist_{g_i}(x, \partial M) \leq K$. By the compactness theorem in, for instance [AT], it follows that a subsequence of (M, g_i) converges in $C^{m,\alpha}$, (and C^∞ in the interior), uniformly on bounded domains containing ∂M, to a limit (M', g). One has $\partial M' = \partial M$ and g is a complete Riemannian metric on M', $C^{m,\alpha}$ up to ∂M and C^∞ in the interior.

By Corollary 5.2, the potential functions u_i also converge in $C^{m,\alpha}$, (in a subsequence) to a limit potential function u on M', and the pair (g, u) gives a solution of the static vacuum Einstein equations. Since $u = \lim u_i$, it follows that

$$u \leq U_0, \tag{5.46}$$

on M', for U_0 as in (5.36). Clearly the boundary metric and mean curvature of (M', g, u) are given by the limit values (γ, H). To prove that $(M', g, u) \in \mathcal{E}^{nd}$, one then needs to prove that (M', g, u) is asymptotically flat and M' is diffeomorphic to M. Note that since the convergence above is only uniform on compact sets, apriori there need not be any relation between the asymptotic structure of (M', g, u) and (M, g_i, u_i) for any given i.

The equation (5.40) holds on each (M, g_i, u_i) and by Corollary 5.2, the constant K is uniform, independent of i. Moreover, as in the proof of Corollary 5.2, there is a geodesic ray $\sigma = \sigma_i$ starting at any fixed base point in $S(1)$ and diverging to infinity, such that on the component $S_c(s)$ of $S(s)$ containing σ, one has

$$\sup_{S_c(s)} |du_i| \leq K s^{-2},$$

where K is independent of i. Since u_i is harmonic, by elliptic regularity, (and scaling), a similar estimate holds for higher derivatives of u_i, and via the static vacuum equations, it follows that

$$\sup_{S_c(s)} |R_{g_i}| \leq Cs^{-3},$$

with again C independent of i. This means that the metrics (M, g_i, u_i) become asymptotically flat at infinity uniformly, at a rate independent of i. For R sufficiently large, $R \geq R_0$ independent of i, the geodesic spheres $S(R)$ and annuli $A(R, 2R)$ are close to Euclidean spheres and annuli, (when scaled by R^{-1}), and hence the geometry is close to that of Euclidean space; there can be no branching or joining of different components of $S(R)$ for $R \geq R_0$. This implies that the limit (M', g, u) has a single asymptotically flat end, and M' is diffeomorphic to M.

■

Remark 5.4. The results above also show that the boundary map Π_B is proper not only on the component $\mathcal{E}^{nd}$ but also its closure $\overline{\mathcal{E}}^{nd}$. In other words, if (M, g_i, u_i) is a sequence of static vacuum solutions in $\mathcal{E}^{nd}$ with boundary data (γ_i, H_i), $(H_i > 0,\, dH_i \neq 0$ on $\{K_i \leq 0\})$, and $(\gamma_i, H_i) \to (\gamma, H)$ in $Met^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$, then (M, g_i, u_i) converges in $C^{m,\alpha}$ (in a subsequence) to a limit (M, g, u) in $\overline{\mathcal{E}}^{nd}$, with $H \geq 0$.

To see this, note that the constant Λ in Theorem 5.1 does not depend on a positive lower bound on H, and so (5.1)-(5.2) hold for the sequence (M, g_i, u_i) above. Of course we are using here the fact that ∂M is outer-minimizing on the sequence (M, g_i, u_i). Similarly in Corollary 5.2, the upper bound U_0 on u_i does not depend on a lower bound for H. One may then use the argument concerning (5.44) to show that u_i cannot go to 0 on M away from ∂M. The proof of Corollary 5.3 also does not require a bound on H away from 0.

It is also worth pointing out that the results of this section only require that ∂M is outer-minimizing in a neighborhood of arbitrarily small but fixed size about ∂M.

6. Degree of Π_B.

By [Sm], a smooth proper Fredholm map $F : B_1 \to B_2$ of Fredholm index 0 between connected Banach manifolds B_1, B_2 has a well-defined degree (mod 2). Namely, if y is a regular value of F, then $F^{-1}(y)$ is a finite set of points, and $deg_{\mathbb{Z}_2} F$ is just the cardinality of $F^{-1}(y)$ (mod 2). In fact if B_1 and B_2 are oriented, then F has a well-defined degree in $\mathbb{Z}$.

By Corollary 5.3, it thus follows that the boundary map

$$\Pi_B : \mathcal{E}^{nd} \to \mathcal{B}^{nd},$$

has a well-defined mod 2 degree, $deg_{\mathbb{Z}_2}\Pi_B$. We do not know if $\mathcal{E}^{nd}$ is orientable, and do not pursue this issue here.

The main result of this section is the following:

Theorem 6.1. *For $(M, \partial M) = (\mathbb{R}^3 \setminus B^3, S^2)$, one has*

$$deg_{\mathbb{Z}_2}\Pi_B = 1. \tag{6.1}$$

Proof: The proof is based on the black hole uniqueness theorem [I], [R], [BM], that the Schwarzschild metrics

$$g_{Sch}(m) = (1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{S^2(1)}, \quad u = \sqrt{1 - \frac{2m}{r}}, \tag{6.2}$$

$r \geq 2m$, are the unique AF static vacuum metrics with a smooth horizon $\mathcal{H} = \{u = 0\}$. Of course the Schwarzschild metrics are not in $\mathcal{E}^{nd}$, but instead lie at the boundary $\partial\mathcal{E}^{nd}$.

Consider for instance any sequence $\{(g_i, u_i)\} \in \mathcal{E}^{nd}$ for which $\Pi_B(g_i, u_i) = (\gamma_i, H_i) \to (\gamma, 0)$ smoothly, with $K_\gamma > 0$. Clearly, $\{(g_i, u_i)\}$ is a divergent sequence in $\mathcal{E}^{nd}$. By Corollary 5.3 and

Remark 5.4, a subsequence of $\{(g_i, u_i)\}$ converges smoothly to a static vacuum limit (M, g, u). (Of course one may have $u = 0$ on ∂M). On this limit,

$$H = 0,$$

at ∂M, so that ∂M is a minimal surface. From (5.6) one has $2\Delta_{\partial M} u = u(|A|^2 + s_\gamma) \geq 0$, and hence it follows from the maximum principle that $u = 0$ on ∂M. Via the static vacuum equations (1.1), this implies further that $A = 0$ and $N(u) = const$ at ∂M. The black hole uniqueness theorem then implies that any such limit is the Schwarzschild metric, and so unique up to scaling. Thus one has uniqueness for the boundary data $(\gamma, 0)$, so that most all boundary metrics γ cannot be realized with $H = 0$ at ∂M, (the no-hair result).

Given this background, suppose

$$deg_{\mathbb{Z}_2} \Pi_B = 0. \tag{6.3}$$

Then for any regular value $(\gamma, H) \in \mathcal{B}^{nd}$, the finite set $\Pi_B^{-1}(\gamma, H)$, if non-empty, consists of at least two distinct static vacuum solutions (g^1, u^1), (g^2, u^2). The regular values of Π_B are open and dense in the range space (by the Sard-Smale theorem). Choose then a sequence of regular values $(\gamma_i, H_i) \to (\gamma_{+1}, 0)$ smoothly. Note that there exist such regular values and corresponding regular points, since there are boundary data $(\gamma, H) \in \Pi_B(\mathcal{E}^{nd})$ arbitrarily close to the Schwarzschild data $(\gamma_{+1}, 0)$; for example, one can go in a bit from the Schwarzschild horizon. (We set $m = 1/2$ here).

Let (g_i^1, u_i^1), (g_i^2, u_i^2) be any pair of corresponding distinct sequences in $\Pi_B^{-1}(\gamma_i, H_i)$. By Corollary 5.3 and Remark 5.4, the sequences (g_i^1, u_i^1), (g_i^2, u_i^2) have $C^{m,\alpha}$ convergent subsequences to limits (g_∞^1, u_∞^1), (g_∞^2, u_∞^2) in $\overline{\mathcal{E}}^{nd}$ and by the uniqueness above

$$g_\infty^1 = g_\infty^2 = g_{Sch}(m),$$

with $m = 1/2$, with $u_\infty^1 = u_\infty^2 = u$ in (6.2).

This implies that near g_{Sch}, the boundary map Π_B is not locally 1-1, and so presumably $D\Pi_B$ has a non-trivial kernel at g_{Sch}. (Note however that $g_{Sch} \notin \mathcal{E}_S$). We claim this is impossible. To prove the claim, let

$$\mathfrak{g}_{Sch} = u^2 d\theta^2 + g_{Sch},$$

be the 4-dimensional Schwarzschild metric on $\mathbb{R}^2 \times S^2$, and similarly let

$$\mathfrak{g}_i^j = (u_i^j)^2 d\theta^2 + g_i^j,$$

be the 4-dimensional static Ricci-flat metrics associated to (g_i^j, u_i^j). By Lemma 2.2, without loss of generality we may assume that each $\mathfrak{g}_i^j$ is in Bianchi gauge with respect to $\mathfrak{g}_{Sch}$, so that, as in (2.10)-(2.11),

$$\beta_{\mathfrak{g}_{Sch}}(\mathfrak{g}_i^j) = 0,$$

for $j = 1, 2$ and i sufficiently large. By the smoothness of the convergence above, one may write

$$\mathfrak{g}_i^j = \mathfrak{g}_{Sch} + \varepsilon_i^j \kappa_i^j + O((\varepsilon_i^j)^2), \tag{6.4}$$

where $L(\kappa_i^j) = 0$ and L is the linearized Einstein operator (2.6) at $\mathfrak{g}_{Sch}$. The data $\mathfrak{g}_i^j$, $\mathfrak{g}_{Sch}$ and κ_i^j are all smooth, (up to the boundary). The forms κ_i^j are only unique up to multiplicative constants, which will be determined by choosing ε_i^j so that the $C^{1,\alpha}$ norm of $\mathfrak{g}_i^j - \mathfrak{g}_{Sch}$ equals ε_i^j. Thus the $C^{1,\alpha}$ norm of κ_i^j is on the order of 1. Note that κ_i^j decays to 0 at infinity, so it is basically supported within compact regions of M. Let $\varepsilon_i = \max(\varepsilon_i^1, \varepsilon_i^2)$. Then

$$\varepsilon_i^{-1}(\mathfrak{g}_i^2 - \mathfrak{g}_i^1) = \kappa_i + O(\varepsilon_i),$$

where $\kappa_i = \varepsilon_i^{-1}(\varepsilon_i^2 \kappa_i^2 - \varepsilon_i^1 \kappa_i^1) \to \kappa$, where the convergence is in $C^{1,\alpha'}$, (in a subsequence). As previously, we need to show that the convergence is strong, so that $\kappa \neq 0$. This follows from a

standard linearization and bootstrap argument, as preceding (5.15). In more detail, dropping the index i, we have $\Delta_{g^j} u^j = 0$, so that

$$\Delta_{g^2}(u^1 - u^2) = (\Delta_{g^2} - \Delta_{g^1})u^1.$$

In local harmonic coordinates, the right side this equation is on the order of ε in $C^{1,\alpha}$, and hence by elliptic regularity, $u_1 - u_2$ is on the order of ε in $C^{3,\alpha}$. Substituting this in the difference of the static equations $u^j Ric_{g^j} = D^2_{g^j} u^j$ and arguing in the same way shows that the difference $g^1 - g^2$ is then also on the order of ε in $C^{3,\alpha}$. This proves the strong convergence.

It follows that the limit form

$$\kappa = (h, u'), \tag{6.5}$$

is a non-zero $C^{1,\alpha}$ weak solution of the linearized static vacuum equations $L(\kappa) = 0$ at g_{Sch} and since $\Pi_B(g_i^1, u_i^1) = \Pi_B(g_i^2, u_i^2)$, one has

$$\gamma_h' = H_h' = 0 \ \text{ at } \ \partial M, \tag{6.6}$$

where $\gamma_h' = h^T = h|_{\partial M}$. As discussed following (5.15), elliptic regularity implies that (h, u') is smooth and so in particular a strong solution. Below we will use the fact that the data (h, u') are in fact real-analytic up to ∂M, (again by elliptic regularity).

We claim that

$$(h, u') = 0 \ \text{ on } \ M, \tag{6.7}$$

which will give a contradiction. This is of course a linearized version of the black hole uniqueness theorem. It is possible that (6.7) can be proved by linearizing one of the existing proofs of black hole uniqueness in [I], [R], [BM]. However, we have not succeeded in doing this and instead (6.7) is proved in a manner similar to the proof of Theorem 5.1.

First, the linearization of (5.6) gives, at ∂M,

$$u' s_\gamma = 2\Delta u'.$$

Since $s_\gamma > 0$, the maximum principle implies that $u' = 0$ at ∂M. Next we claim $A' = 0$. To see this, the vacuum equations give $uRic = D^2 u$ and $D^2 u = N(u)A + (D^2 u)^T$ when evaluated on tangent vectors to ∂M. Taking then the variation and evaluating tangentially gives $(uRic)' = 0$ so $0 = (D^2 u)' = (D^2)' u + D^2 u'$. The first term on the right vanishes when evaluating tangentially and hence so does the second term. This implies $0 = (N(u)A)' = N(u)A' + N(u')A = N(u)A'$. Since $N(u) = const \neq 0$, it follows that $A' = 0$. Similarly, taking the variation of the divergence or vector constraint gives $N(u') = const$.

Clearly $N(u') = m'$, (up to constants). A simple examination of the proof of black hole uniqueness in [R] applied to an Einstein deformation as in (6.5) and satisfying (6.6), shows easily that $N(u') = 0$ at ∂M. (One does not obtain any further information, since the bulk data in the Robinson proof, via divergence identities, are quadratic in the deviation from Schwarzschild).

Thus the variations $(\gamma', u', A', N(u'))$ of all the Cauchy data are trivial. As in the proof of Theorem 5.1, we use a bootstrap argument to prove that the data (h, u') vanish to infinite order at ∂M, in geodesic gauge.

Thus, using geodesic normal coordinates near ∂M, write

$$g = dt^2 + g_t,$$

where $t(x) = dist(x, \partial M)$. We may assume, (by adding an infinitesimal deformation of the form $\delta^* V$ if necessary), that h preserves this gauge, so that $h_{0\alpha} = 0$, i.e. $h(N, \cdot) = 0$, $N = \partial_t$, near ∂M. By the discussion above, we have $u' = N(u') = 0$ at ∂M and similarly $h = \nabla_N h = 0$ at ∂M, so that

$$u' = O(t^2) \ \text{ and } \ h = O(t^2). \tag{6.8}$$

The variation of the potential equation $\Delta_M u = 0$ gives

$$\Delta u' = -\Delta' u = \langle D^2 u, h\rangle - \langle \beta(h), du\rangle, \tag{6.9}$$

where β is the Bianchi operator, (cf. [Be]). Since $\beta(h) = 0$ at ∂M, this gives $\Delta u' = 0$ at ∂M and hence $NN(u') = 0$ at ∂M, so that

$$u' = O(t^3). \tag{6.10}$$

Next the linearization of the Riccati equation gives $(\nabla_N A)' = -(A^2)' - (R_N)' = -(R_N)'$ at ∂M. One computes $*R_N = -(Ric^T) = -u^{-1}(D^2u)^T$, so $(*R_N)' = u^{-2}u'D^2u - u^{-1}(D^2)'u - u^{-1}D^2u'$, as a form on $T(\partial M)$. It follows from (6.10) that $(*R_N)' = 0$ at ∂N and hence $(\nabla_N A)' = 0$ so that

$$h = O(t^3). \tag{6.11}$$

Next taking the normal derivative of (6.9) gives

$$\Delta N(u') = \langle \nabla_N D^2 u, h\rangle + \langle D^2 u, \nabla_N h\rangle + \langle \nabla_N \beta(h), du\rangle + \langle \beta(h), \nabla_N du\rangle,$$

which vanishes at ∂M and hence $u' = O(t^4)$. Substituting this in the linearized Riccati equation above and using previous estimates gives $h = O(t^4)$, and so on. It follows that (h, u') vanish to infinite order at ∂M. Since (h, u') are real-analytic up to ∂M, (in geodesic gauge), this implies that $h = u' = 0$ on M, i.e. (6.7) holds. This completes the proof.

■

Although the proof of Theorem 6.1 implies that $D\Pi_B$ has trivial kernel at the Schwarzschild metric g_{Sch}, one does not expect this to be the case for the cokernel. In fact, one expects that $Coker D\Pi_B$ is infinite dimensional, in that any boundary variation of the form $(k, 0)$, where k is a variation of the boundary metric, is not tangent to a curve of static metrics with $H = 0$ at ∂M. This amounts to the linearized version of the no-hair theorem, (which has not been proved as far as we are aware). In particular, we expect $D\Pi_B$ is not Fredholm at g_{Sch}.

Remark 6.2. We point out that if $\mathcal{E}^{nd}$ is orientable, then the proof above shows that $deg_{\mathbb{Z}}\Pi_B = \pm 1$, depending on the choice of orientation of $\mathcal{E}^{nd}$, (and the target space).

Remark 6.3. It is interesting to compare Theorem 6.1 with the bounded domain case. Thus suppose $(M, \partial M) = (B^3, S^2)$ and consider the space $\mathcal{E} = \mathcal{E}^{m,\alpha}$ of $C^{m,\alpha}$ static vacuum solutions on M. This is again a smooth Banach manifold for which the boundary map $\Pi_B(g) = (\gamma, H)$ is Fredholm, of index 0. By general theory, Π_B thus always has a local degree.

We claim that the local degree of Π_B near the standard flat metric solution $(g, u) = (g_{flat}, 1)$ on $B^3(1)$ is 0,

$$deg_{\mathrm{loc}}\Pi_B = 0.$$

Thus, consider static vacuum metrics (g, u) near the flat data, with induced boundary data (γ, H). Since $K_\gamma > 0$, there is a unique isometric embedding $(\partial M, \gamma) \to \mathbb{R}^3$, with induced mean curvature H_0. By a basic result of Shi-Tam [ST],

$$\int_{\partial M} H dV_\gamma \leq \int_{\partial M} H_0 dV_\gamma,$$

so that the functional $\int_{\partial M}(H - H_0)dV_\gamma$ on $\mathcal{E}$ has a local maximum at $(g, u) = (g_{flat}, 1)$. In particular, the boundary data (γ_{+1}, H), for any H with $H > 2$, is not in $Im\Pi_B$, so that Π_B is not locally surjective. Hence, the local degree is 0.

It also follows that the solution $(g_{flat}, 1)$ on $B^3(1)$ is a critical point of Π_B. In fact there are curves (g_t, u_t) of static vacuum solutions with $(g_0, u_0) = (g_{flat}, 1)$ and $\Pi_B(g_t) = (\gamma_{+1}, 2) = \Pi_B((g_{flat}, 1))$. For this, one can just take $g_t = g_{flat}$ and $u_t = 1 + tz$, where z is any affine function on $\mathbb{R}^3$.

Similarly, the main result in [Mi] also shows that the standard flat data $(g_{flat}, 1)$ on $\mathbb{R}^3 \setminus B^3(1)$, with boundary data $(\gamma_{+1}, 2)$ is a critical point of Π_B. Thus, the standard flat boundary data $(\gamma_{+1}, 2)$ is not a regular value of Π_B, for either the interior or exterior boundary map Π_B.

As noted in the Introduction, Theorem 6.1 implies Theorem 1.1, since any smooth proper Fredholm map of non-zero degree is surjective.

We conclude this work with several remarks and questions worthy of further study. First, it would be interesting to know when solutions (M, g) of the static vacuum Einstein equations (1.1) are uniquely determined, (up to isometry), by the Bartnik boundary data (γ, H). This is unknown for solutions in $\mathcal{E}^{nd}$; in fact it is even unknown for solutions close to the standard solution $(\mathbb{R}^3 \setminus B^3(1), g_{flat}, 1)$, cf. Remark 6.3 above.

Note that the mass m of the solutions in $\mathcal{E}^{nd}$ is not necessarly non-negative, but can take on all values in $\mathbb{R}$. It is an interesting question to determine conditions on the data (γ, H) for which the corresponding solution(s) (M, g) have $m > 0$ or $m \geq 0$; cf. [H, Prop.2.1] for some results on this issue. For static vacuum solutions, the mass m can be computed concretely at ∂M as

$$m = \frac{1}{4\pi}\int_{\partial M} N(u) dA_\gamma.$$

Finally, recall that Bray [Br] has suggested replacing the "no horizon" condition in the definition of admissible extension in (1.5) by the property that ∂M is outer-minimizing in (M, g). Conceptually, this meshes well with the results here, (since all solutions in $\mathcal{E}^{nd}$ have outer-minimizing boundary). However, as discussed previously, it is not easy to determine from the Bartnik boundary data alone which solutions have outer-minimizing boundary.

7. Appendix.

Here we derive the first variation of energy formula (4.7)-(4.8) used to obtain the bounds for the mean curvature for the generalized IMCF in §4. The first task is to derive the appropriate evolution equation. Recall that the flow is given by $\partial_t = (wH)^{-1}N$; we will use the notation of §4. Therefore

$$\begin{aligned} \partial_t H &= -\overline{\Delta}(wH)^{-1} - (wH)^{-1}(|A|^2 + R_{NN}) \\ &= (wH)^{-2}\overline{\Delta}(wH) - 2(wH)^{-1}|\overline{\nabla}\log(wH)|^2 - (wH)^{-1}(|A|^2 + R_{NN}), \end{aligned}$$

and, since u is harmonic on (M, g),

$$wH^2\partial_t u = -\overline{\Delta}u - R_{NN}u.$$

Let $P(u)$ be a positive function of the potential. Then

$$\begin{aligned} (7.1)\quad \partial_t(wHP) = \; & wP[(wH)^{-2}\overline{\Delta}(wH) - 2(wH)^{-1}|\overline{\nabla}\log(wH)|^2 \\ & -(wH)^{-1}(|A|^2 + R_{NN}) + w^{-1}N(\log w) - (wH)^{-1}(\log P)'(\overline{\Delta}u + R_{NN}u)]. \end{aligned}$$

Now consider

$$\begin{aligned} & f^{-\lambda}\partial_t \int_{\Sigma_t} \psi(x)(f(t)wHP(u)G(u))^\lambda \\ = & \int_{\Sigma_t} \lambda\psi(wHP)^{\lambda-1}G^\lambda\partial_t(wHP) + \lambda(\log f)'\psi(wHPG)^\lambda \\ & + \; \lambda\psi(wHPG)^\lambda(wH)^{-1}(\log G)'uN(\log u) + \psi(wHPG)^\lambda w^{-1}, \end{aligned}$$

where $G(u)$ is another positive function of the potential, to be chosen below. According to (7.1) there are two terms which should be integrated by parts, namely

$$\int_{\Sigma_t} \psi(x)(wHP)^{\lambda-1} G^\lambda wP(wH)^{-2}\overline{\Delta}(wH) = -\int_{\Sigma_t} \psi w(PG)^\lambda (wH)^{\lambda-2} Q_1,$$

where

$$\begin{aligned} Q_1 &= (\lambda-3)|\overline{\nabla}\log(wH)|^2 + \langle \overline{\nabla}\log(w\psi), \overline{\nabla}\log(wH)\rangle \\ &\quad +\lambda(\log P)' u\langle \overline{\nabla}\log u, \overline{\nabla}\log(wH)\rangle + \lambda(\log G)' u\langle \overline{\nabla}\log u, \overline{\nabla}\log(wH)\rangle, \end{aligned}$$

and

$$-\int_{\Sigma_t} \psi(x)(wHP)^{\lambda-1} G^\lambda PH^{-1}(\log P)'\overline{\Delta}u = \int_{\Sigma_t} \psi w(PG)^\lambda (wH)^{\lambda-2} Q_2,$$

where

$$\begin{aligned} Q_2 &= (\lambda-1)(\log P)' u\langle \overline{\nabla}\log(wH), \overline{\nabla}\log u\rangle + ((\lambda-1)P'^2P^{-2} + P''P^{-1})u^2|\overline{\nabla}\log u|^2 \\ &\quad -(\log P)' u\langle \overline{\nabla}\log u, \overline{\nabla}\log(wH)\rangle + (\log P)' u\langle \overline{\nabla}\log u, \overline{\nabla}\log w\rangle \\ &\quad +\lambda(\log G)'(\log P)' u^2|\overline{\nabla}\log u|^2 + (\log P)' u\langle \overline{\nabla}\log\psi, \overline{\nabla}\log u\rangle. \end{aligned}$$

We now have

$$f^{-\lambda}\partial_t \int_{\Sigma_t} \psi(x)(f(t)wHP(u)G(u))^\lambda = \int_{\Sigma_t} \lambda\psi w(PG)^\lambda (wH)^{\lambda-2} Q_3,$$

where

$$\begin{aligned} Q_3 &= -(\lambda-1)|\overline{\nabla}\log(wH)|^2 - (2(\log P)' + \lambda(\log G)')u\langle \overline{\nabla}\log u, \overline{\nabla}\log(wH)\rangle \\ &\quad +((\log P)'' + \lambda(\log P)'^2 + \lambda(\log G)'(\log P)')u^2|\overline{\nabla}\log u|^2 - |A|^2 + \lambda^{-1}H^2 \\ &\quad -(1+(\log P)'u)R_{NN} + w(\log f)'H^2 + HN(\log w) + H(\log G)'uN(\log u) \\ &\quad -\langle \overline{\nabla}\log(w\psi), (\overline{\nabla}\log(wH) - (\log P)'u\overline{\nabla}\log u)\rangle. \end{aligned}$$

In order that the quadratic form involving first derivatives of $\log(wH)$ and $\log u$ be definite, we choose $G = P^{-2}$. Then upon completing the square the quadratic form becomes

$$((\log P)'' - (\log P)'^2)u^2|\overline{\nabla}\log u|^2 - (\lambda-1)|\overline{\nabla}\log(wH) - (\log P)'u\overline{\nabla}\log u|^2.$$

Lastly setting $F = P^{-1}$ yields the desired expression (4.7)-(4.8).

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Dept. of Mathematics, S.U.N.Y. at Stony Brook, Stony Brook, NY 11790
E-mail address: `anderson@math.sunysb.edu`
E-mail address: `khuri@math.sunysb.edu`